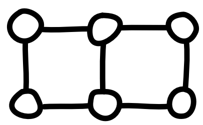
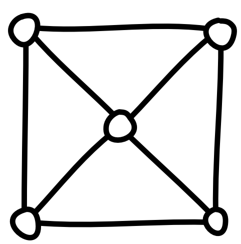
<u>Graphs</u>

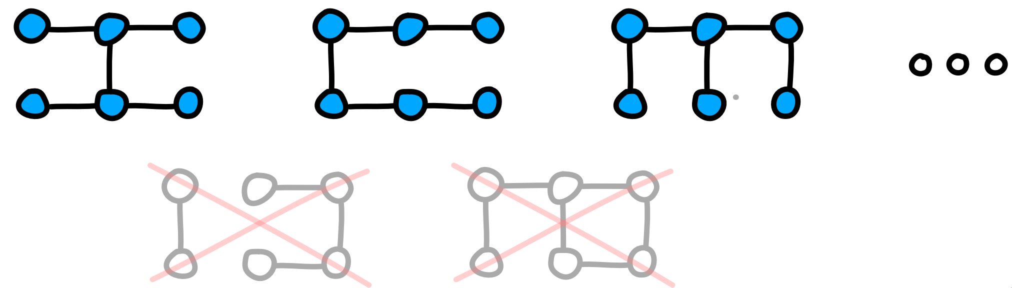


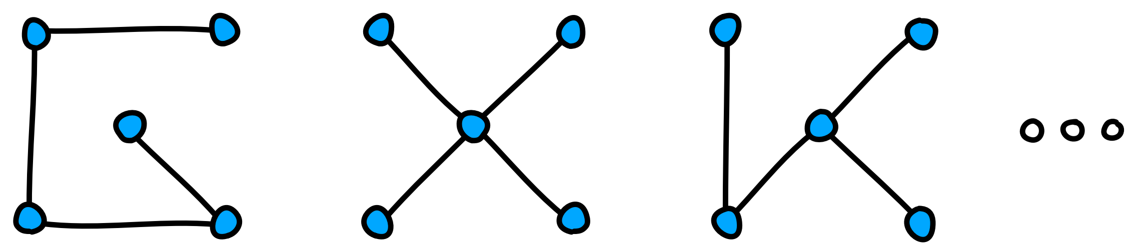
2×3 grid graph



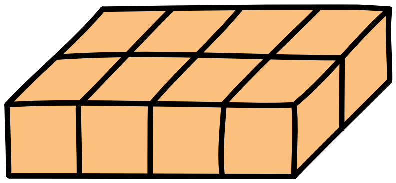
(4 + 1) wheel graph

<u>Spanning Trees</u>

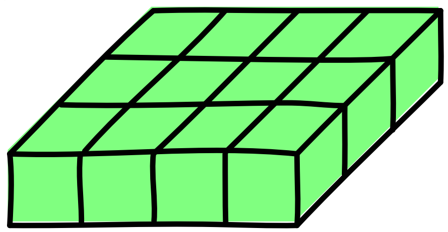




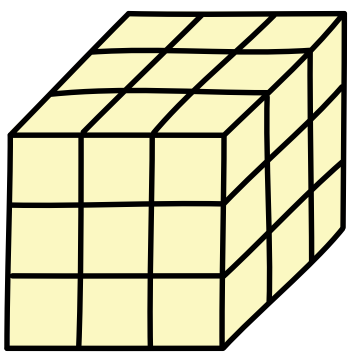
connected
$E = V - 1$
no cycles



2×4 (×1)



3×4×1



3×3×3

Snap Cube Structure:

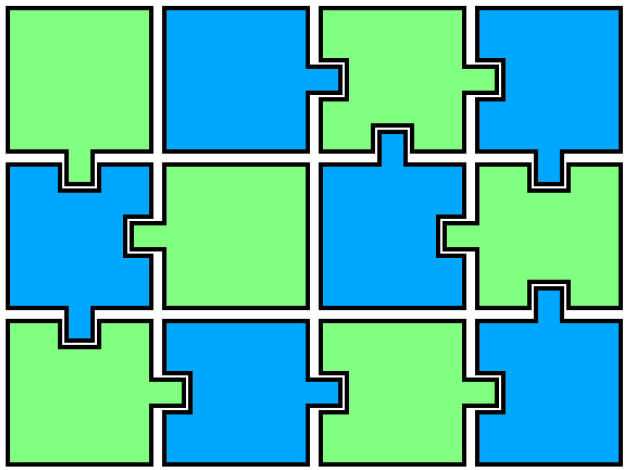
[connected,

smooth

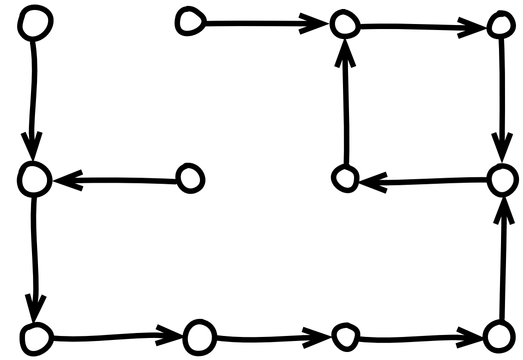
orientation of the posts]

How many structures

- on the 3×3×3 cube?
- on other cubes & bricks?

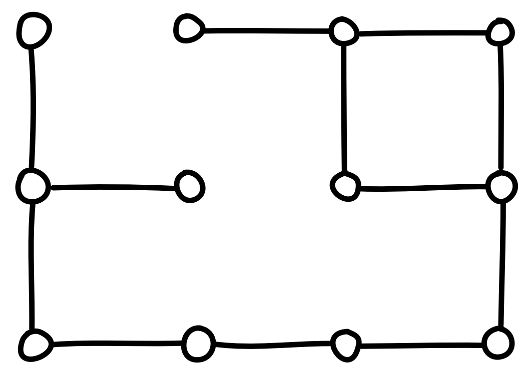


3×4×1
snap cube structure



directed subgraph of
3×4 grid graph

?

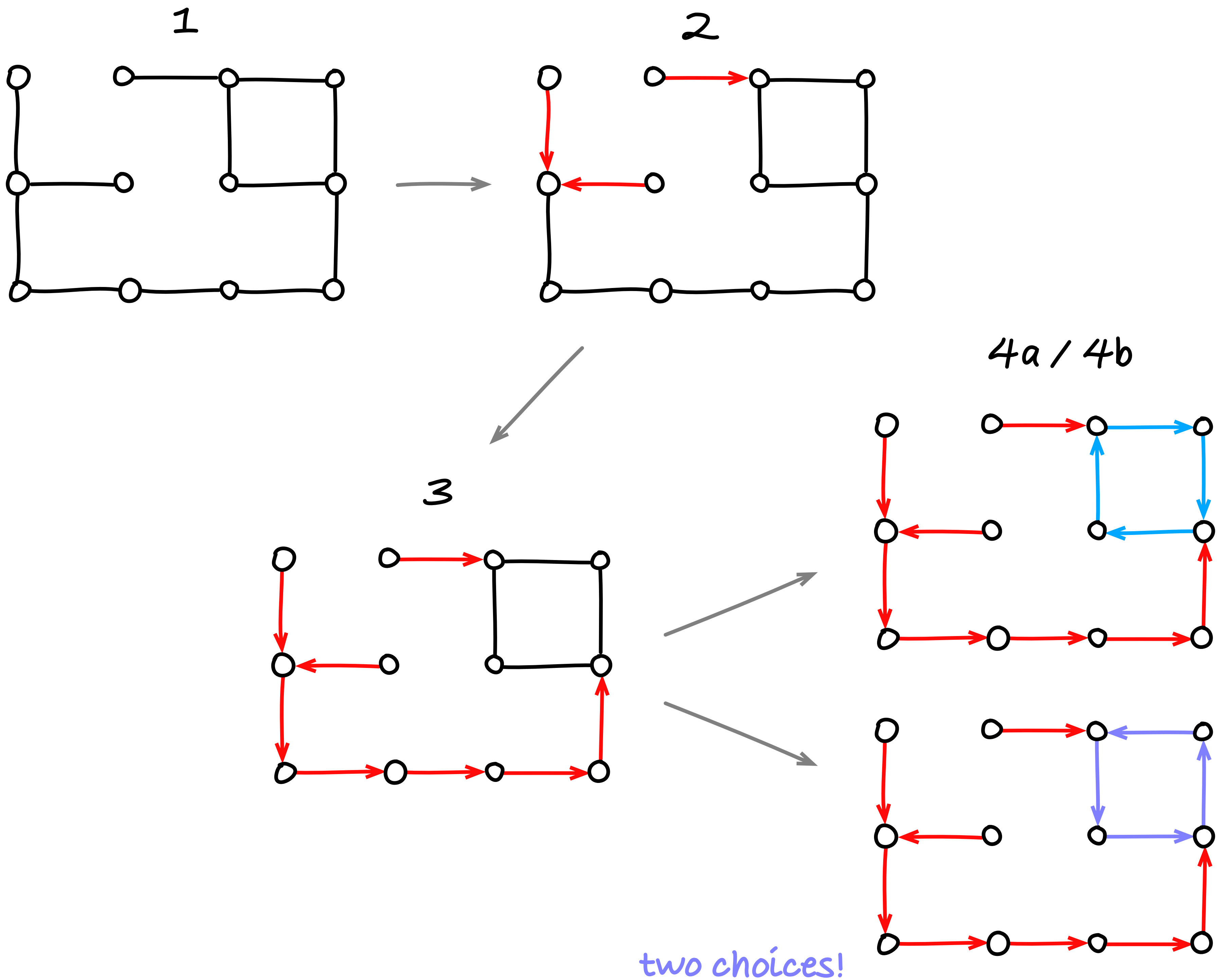


connected

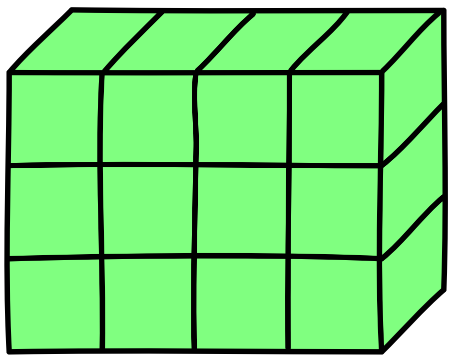
V = E

1 cycle

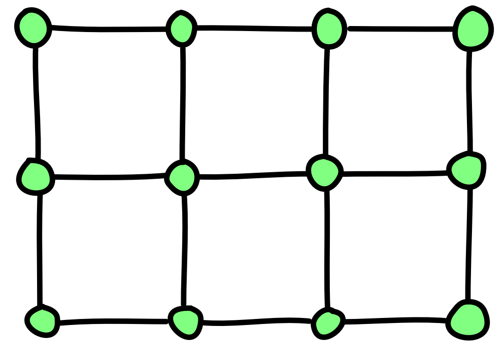
undirected subgraph


1
2
3
4a / 4b
two choices!

Correspondence #1



2
snap-cube
structures

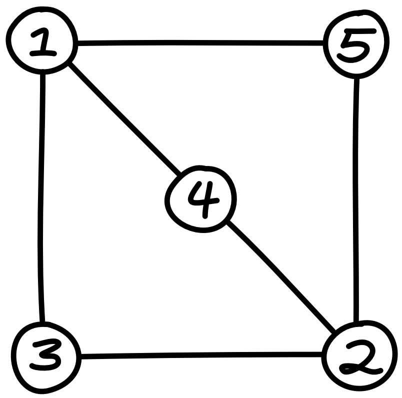


1
spanning
unicyclic
subgraph

"unicycle"

(= spanning tree + 1 edge)

Graphs to matrices



adjacency matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

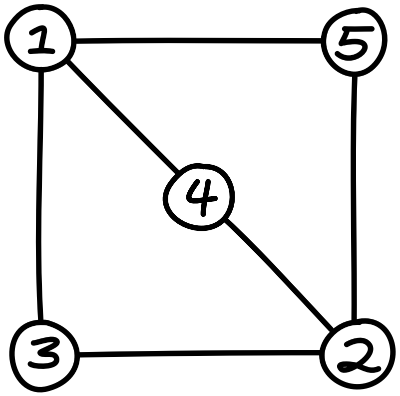
degree matrix

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Laplace matrix

$$\begin{bmatrix} -3 & 0 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}$$

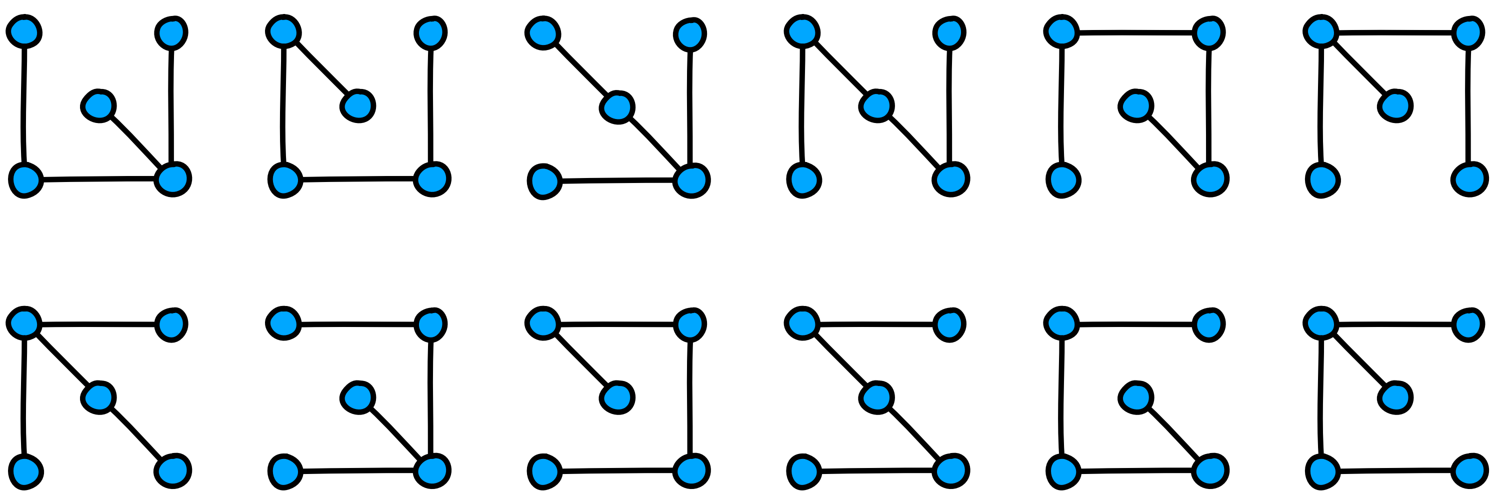
Kirchhoff's Theorem



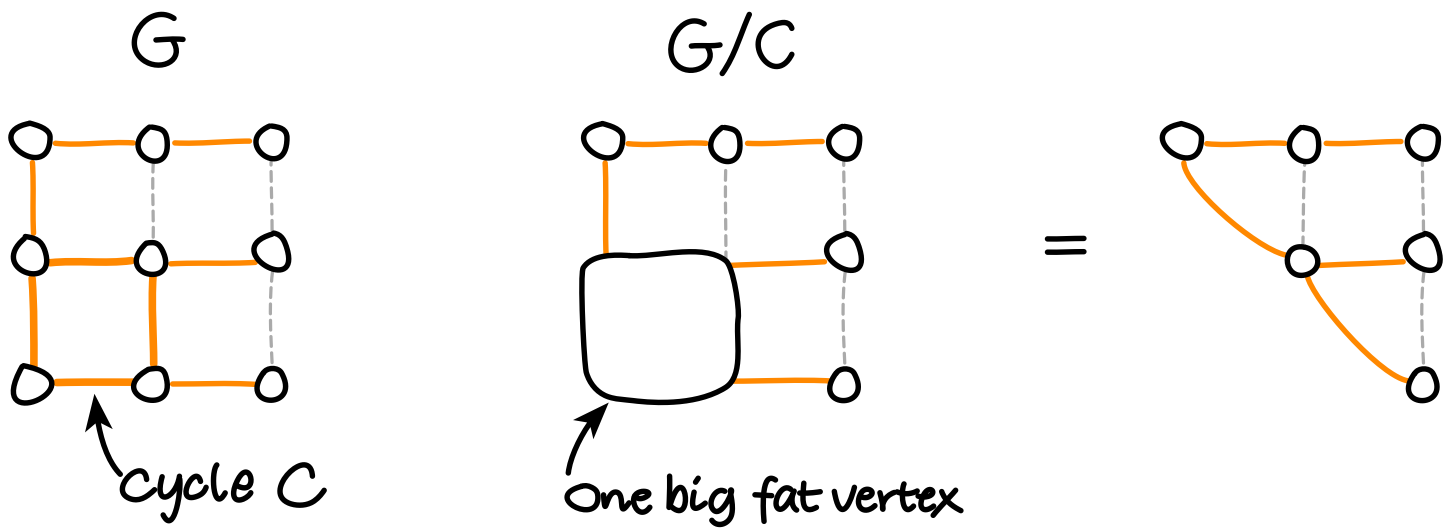
any
Laplace minor

$$\begin{bmatrix} \cancel{-3} & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{0} & -3 & 1 & 1 & 1 \\ \cancel{1} & 1 & -2 & 0 & 0 \\ \cancel{1} & 1 & 0 & -2 & 0 \\ \cancel{1} & 1 & 0 & 0 & -2 \end{bmatrix}$$

$$\det \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} = 12$$

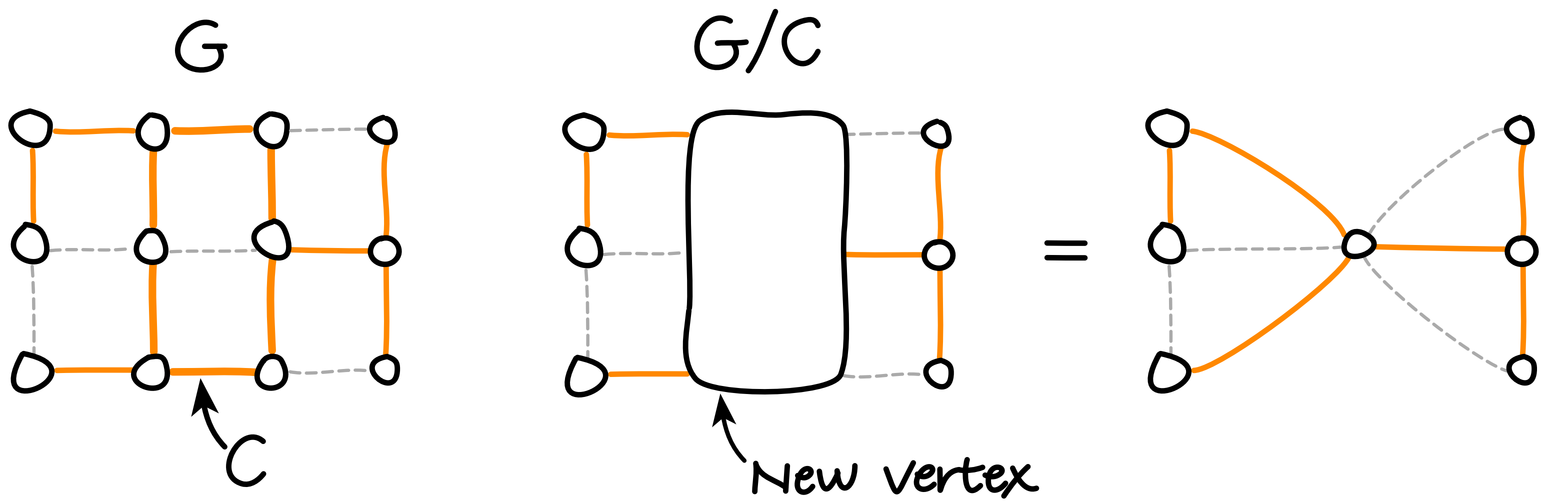


Ex 1

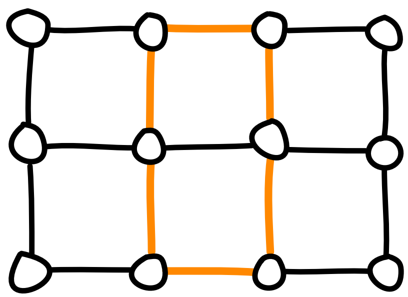


unicycle in G $\rightleftarrows$ tree in G/C

Ex 2

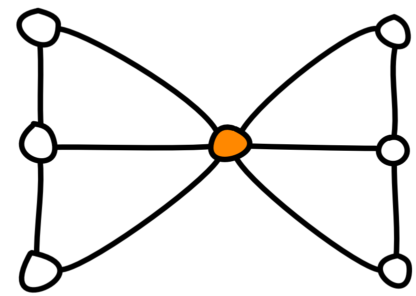


Correspondence #2

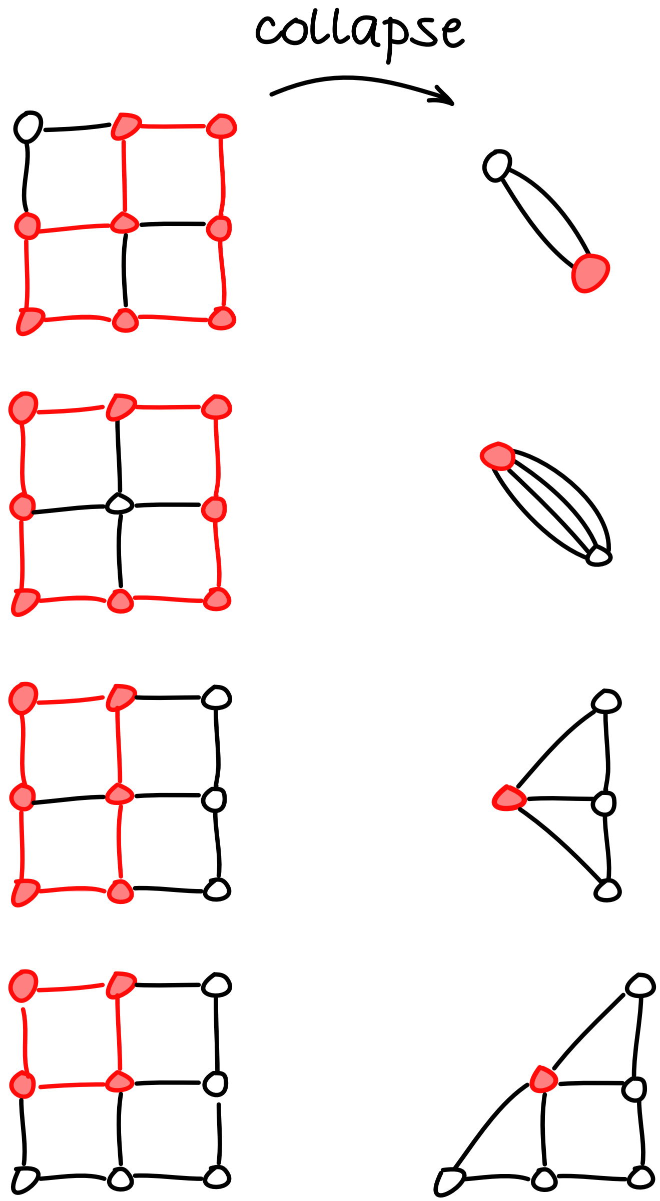


unicycles with cycle C

one-to-one



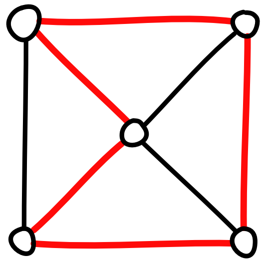
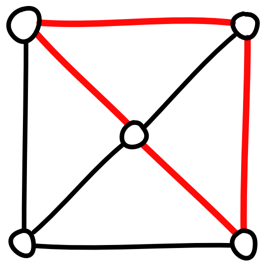
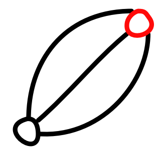
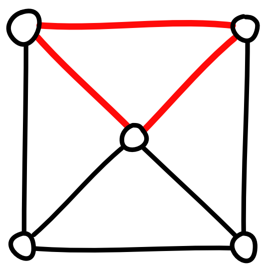
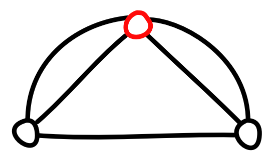
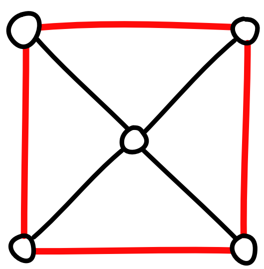
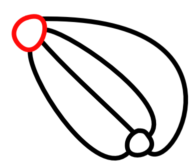
spanning trees in G/C

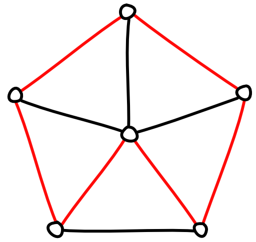
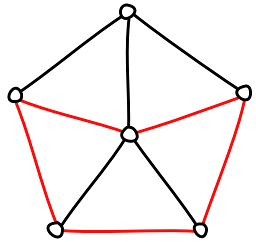
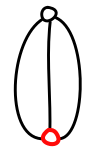
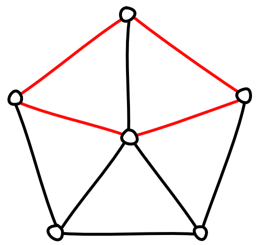
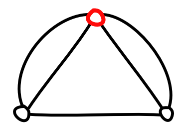
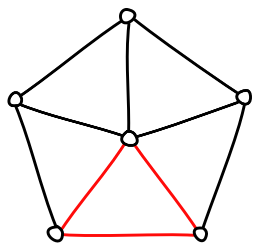
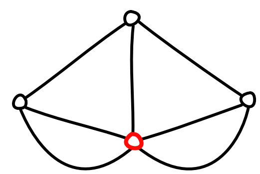
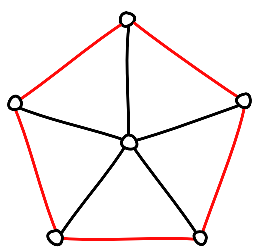
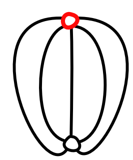


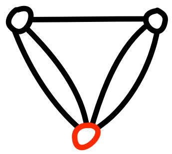
trees × symmetries	
2 × 4	8
4 × 1	4
8 × 4	32
Kirchhoff!	
30 × 4	120

Total unicycles 164

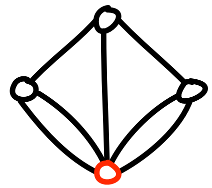
Total snap-cube structures 328 !

Hub+4			1 x 4	4
Hub+3			3 x 4	12
Hub+2			8 x 4	32
Rim			4 x 1	4
			Total unicycles	52

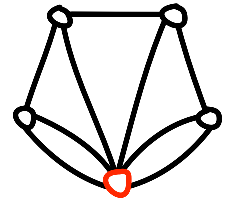
Hub+5			1 x 5	5
Hub+4			3 x 5	15
Hub+3			8 x 5	40
Hub+2			21 x 5	105
			Kirchhoff	
Rim			5 x 1	5
			Total unicycles	170



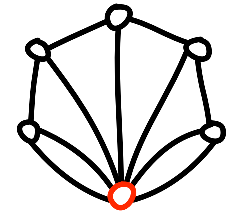
H_3



H_4



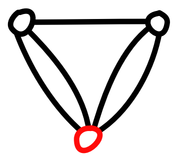
H_5



H_6

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

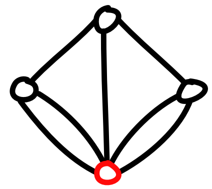
8 21 55 144 ...



H_3

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

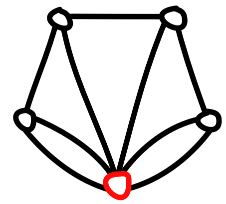
8



H_4

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

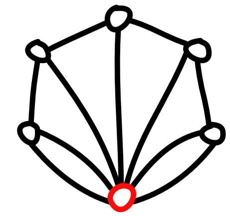
21



H_5

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

55



H_6

$$\begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

144

...

---

induction...

(Prop) $\boxed{T(H_n) = F_{2n}}$

Unicycles on the $(n+1)$ Wheel

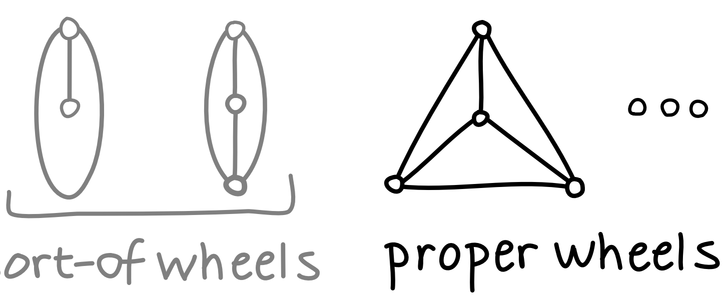
$$U_n = \overbrace{n \cdot 1}^{\text{rim}} + \overbrace{\sum_{k=1}^{n-1} n \cdot F_{2k}}^{\text{hub}+k}$$

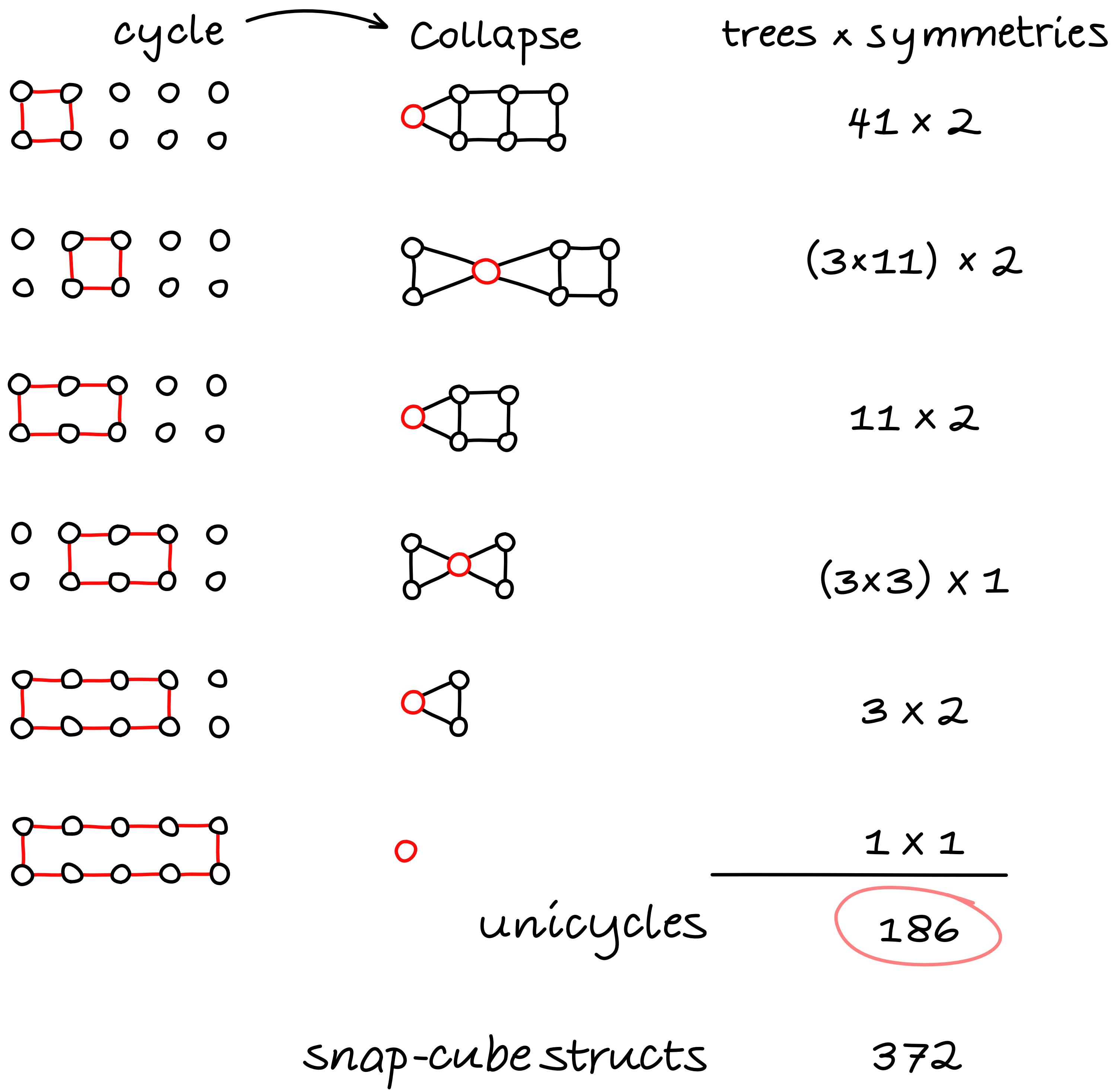
$$= n \cdot \left[1 + \sum_{k=1}^{n-1} F_{2k} \right]$$

routine induction

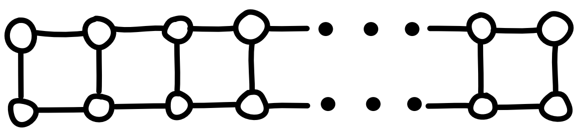
$$= n \cdot F_{2n-1}$$

n	1	2	3	4	5	6	7	8	...
U_n	1	4	15	52	170	534	1631	4880	

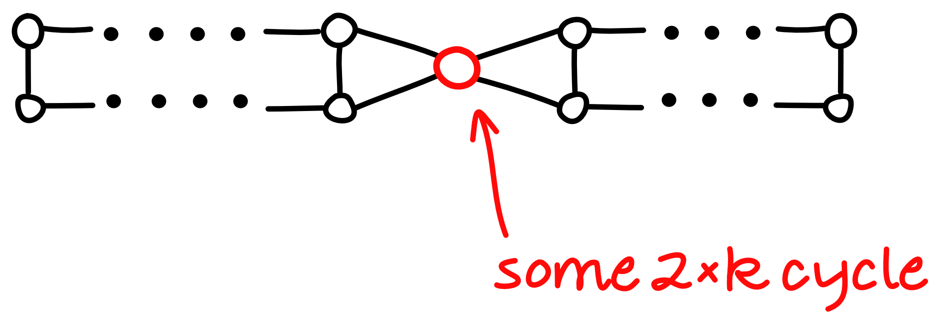


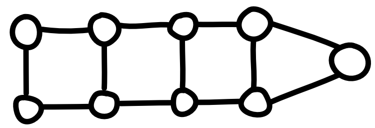

cycle
Collapse
trees x symmetries
41 x 2
(3x11) x 2
11 x 2
(3x3) x 1
3 x 2
1 x 1
unicycles
186
snap-cube structs
372

2xn grid



typical collapse



$T_r = T\Big[$  $\Big]$

r pairs

fun!

1. (Prop) $\boxed{T_r = 4T_{r-1} - T_{r-2}}$ $\quad (T_0 = 1,\ T_1 = 3)$

$$\left(\Rightarrow \quad T_r = (3+\sqrt{3})^{2r+1} / 6^{r+1} \right)$$

2. (Prop) $\boxed{U_n = \sum_{r=0}^{n-2} \sum_{k=2}^{n-r} T_r \times T_{n-k+r}}$

Unicycles on the $2 \times n$ grid

$$T_r = 4T_{r-1} - T_{r-2} \qquad U_n = \sum_{r=0}^{n-2} \sum_{k=2}^{n-r} T_r \times T_{n-k+r}$$

n	1	2	3	4	5	6	7	8	...
U_n	0	1	7	38	186	859	3821	16556	

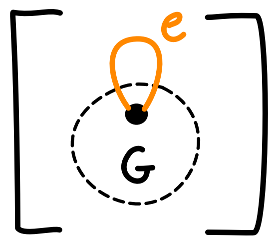
3. (Prop) On the $2 \times n$ grid,

$$\boxed{U_n = 8U_{n-1} - 18U_{n-2} + 8U_{n-3} - U_{n-4}}$$

fwiw

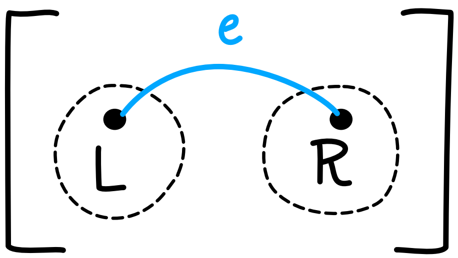
U-T Relations:

Loop rule

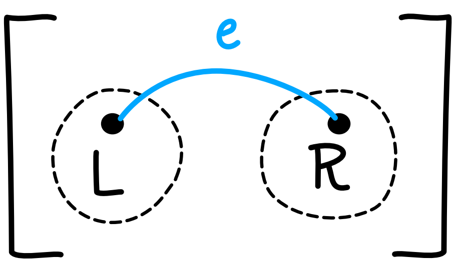
U 

$= U[G-e] + T[G-e]$

Bridge rule / Product rule

T 

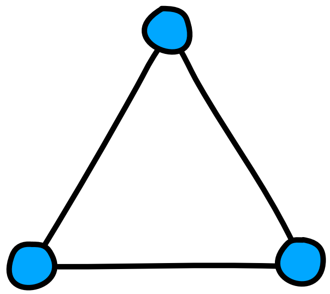
$= T[L]\,T[R]$

U 

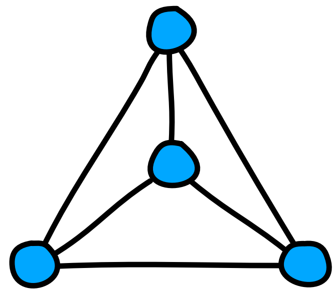
$= U[L]\,T[R] + U[R]\,T[L]$

calculus?!

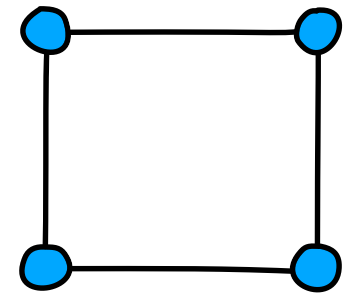
Tutte Polynomials



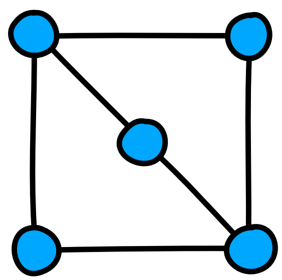
$x^2 + x + y$



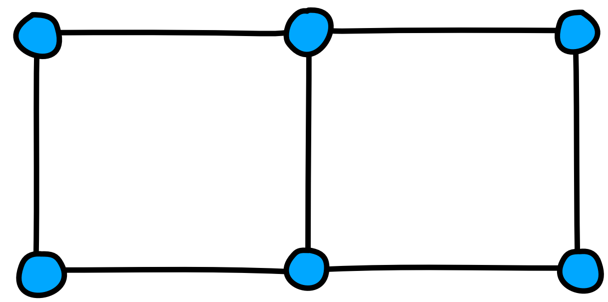
$$x^3 + 3x^2 + 4xy + 2x + y^3 + 3y^2 + 2y$$



$x^3 + x^2 + x + y$

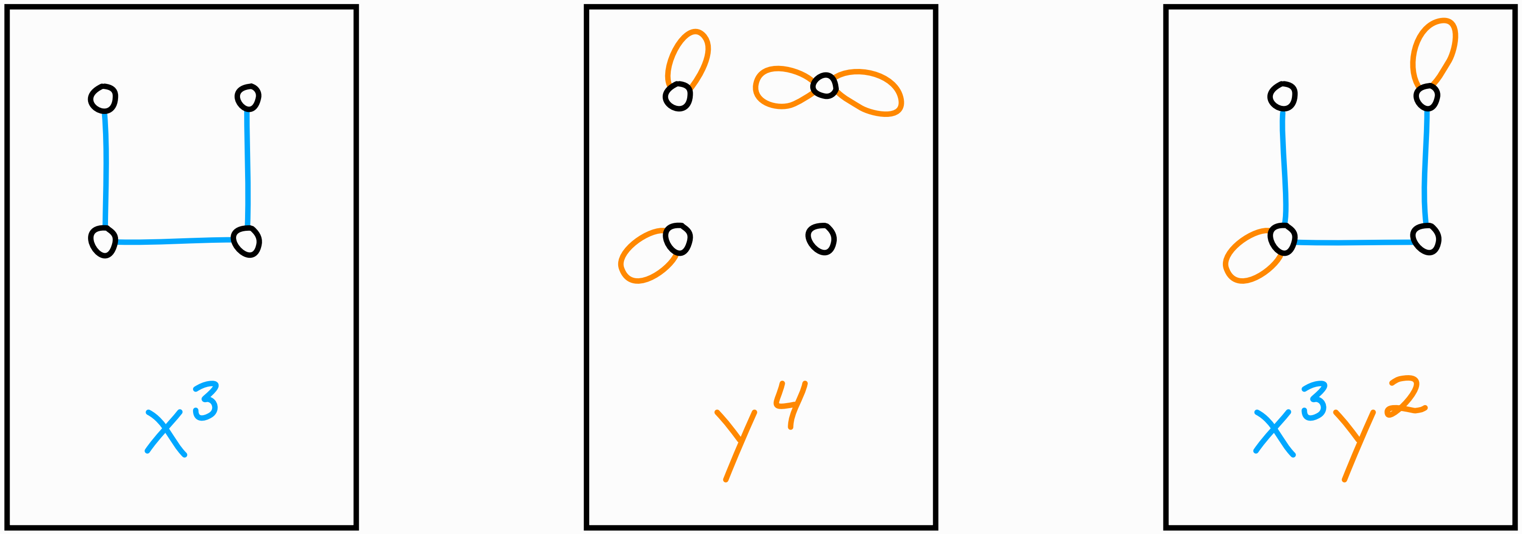


$$x^4 + 2x^3 + 3x^2 + 3xy + x + y^2 + y$$



$$x^5 + 2x^4 + 3x^3 + 2x^2y + 2x^2 + 2xy + x + y^2 + y$$

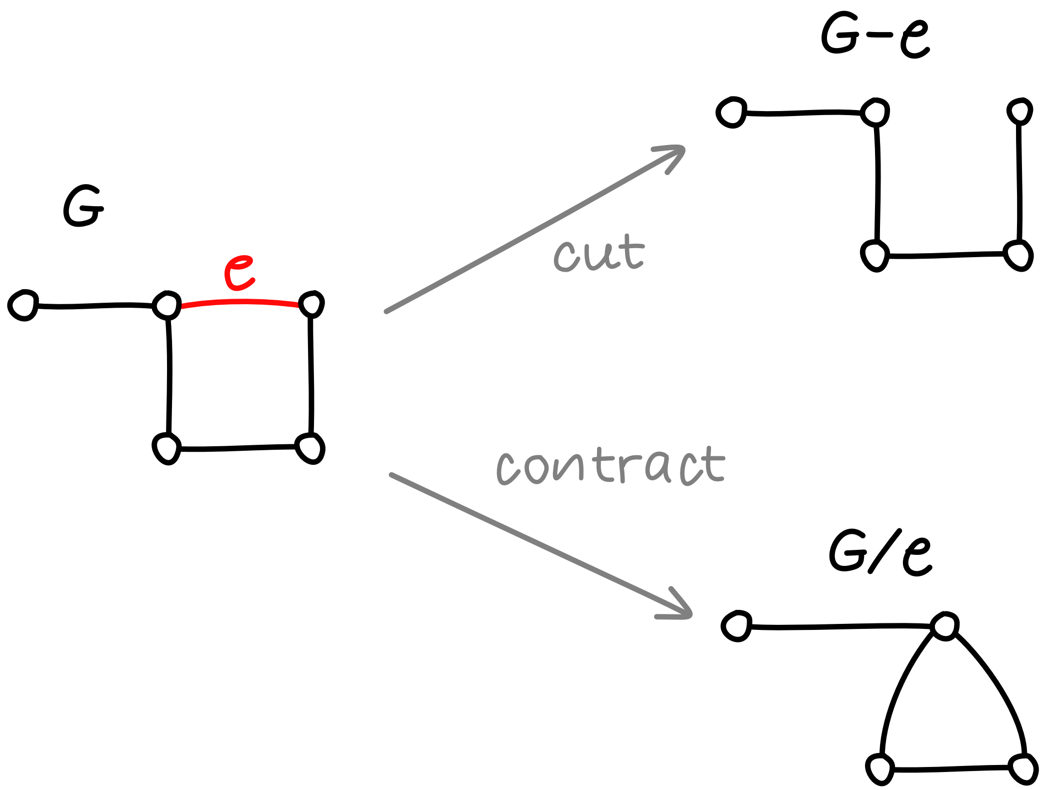
Base cases



$$\left.\begin{array}{l}\text{i bridges}\\ \text{j loops}\end{array}\right\} \longmapsto x^i y^j$$

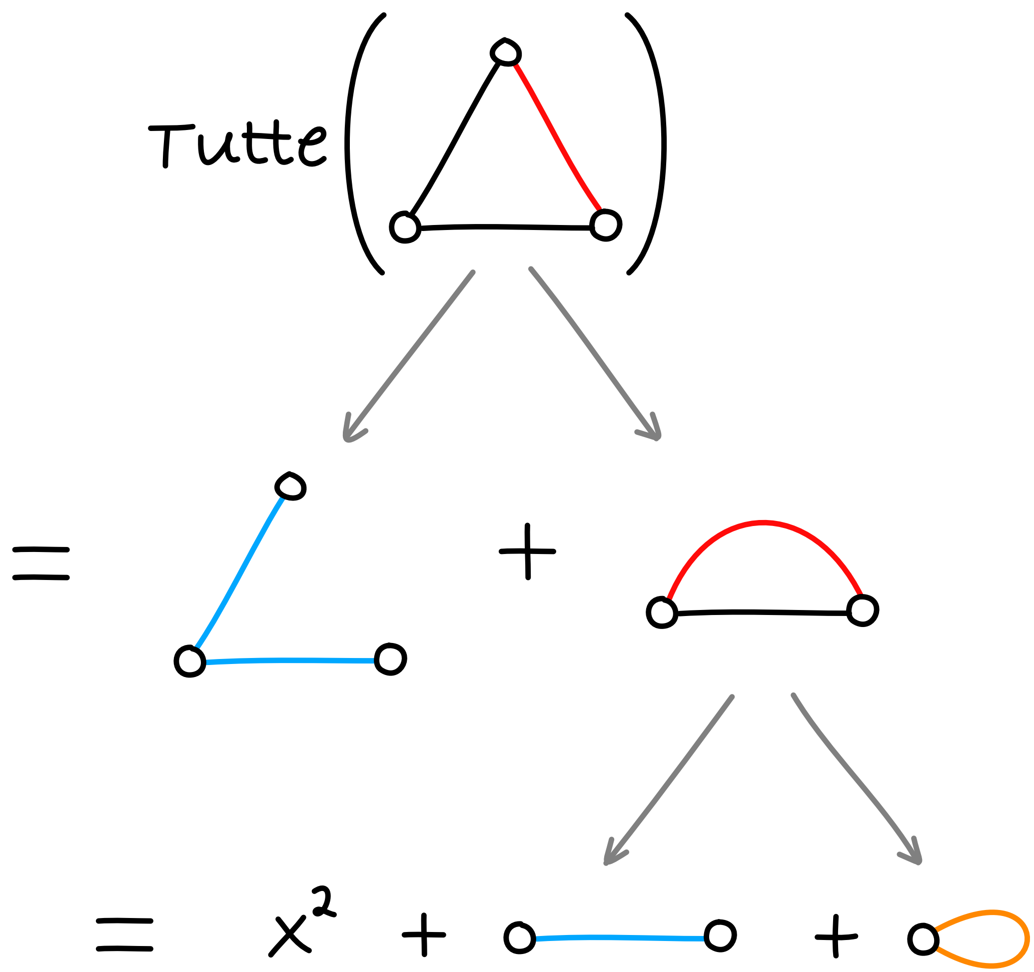
ordinary edges

(not a bridge, not a loop)



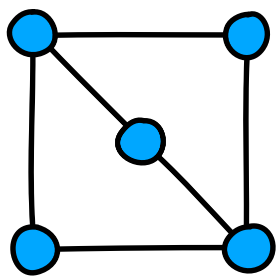
$$T_G(x,y) = T_{G-e}(x,y) + T_{G/e}(x,y)$$

Ex.



$$= x^2 + x + y.$$

(Prop) $\quad \mathrm{Trees}(G) = \mathrm{Tutte}_G(1,1).$

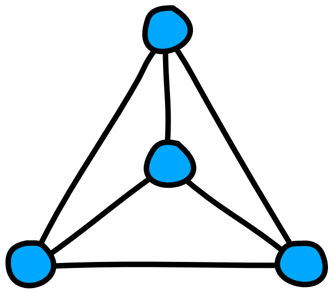


$$x^4 + 2x^3 + 3x^2 + 3xy + x + y^2 + y$$

$$\Big\downarrow \; \begin{array}{l} x \to 1 \\ y \to 1 \end{array}$$

$$1 + 2 + 3 + 3 + 1 + 1 + 1 = 12$$

$$(\text{Prop}) \quad \mathcal{U}(G) = \frac{\partial}{\partial y}\left[\mathrm{Tutte}_G(1,y)\right]_{y=1}.$$



$$x^3 + 3x^2 + 4xy + 2x + y^3 + 3y^2 + 2y$$

$$\Big\downarrow \; x \to 1$$

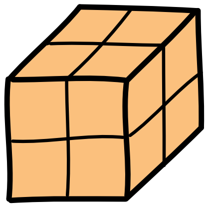
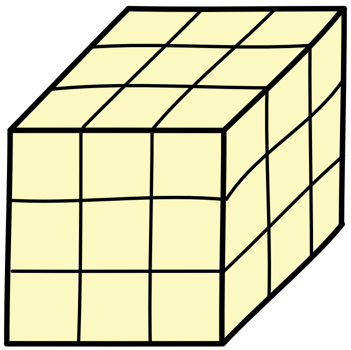
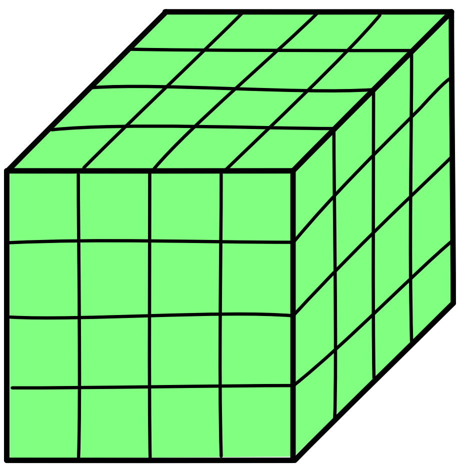
$$1 + 3 + 4y + 2 + y^3 + 3y^2 + 2y$$

$$\Big\downarrow \; \frac{\partial}{\partial y}$$

$$4 + 3y^2 + 6y + 2$$

$$\Big\downarrow \; y \to 1$$

$$15$$

	Cube	Snap-cube structures
	2x2x2	816 (exact - via Tutte poly)
	3x3x3	74,571,238,440,960 (exact - via Tutte poly)
	4x4x4	3.7×10^{36} or so (probabilistic estimate)

Read!

Counting on determinants
A.T. Benjamin, N.T. Cameron
The American Mathematical Monthly 112(6), 481–492

A.T. Benjamin, C.R. Yerger
Combinatorial interpretations of spanning tree identities
Bull. Inst. Combin. Appl., 47 (2006), pp. 37–42

Approach to criticality in sandpiles
Anne Fey, Lionel Levine, and David B. Wilson
Phys. Rev. E 82, 031121 (2010)