A Modern Treatment of the 15 Puzzle

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1. INTRODUCTION. In the 1870's the impish puzzlemaker Sam Loyd caused quite a stir in the United States, Britain, and Europe with his now-famous 15-puzzle. In its original form, the puzzle consists of fifteen square blocks numbered 1 through 15 but otherwise identical and a square tray large enough to accommodate 16 blocks. The 15 blocks are placed in the tray as shown in Figure 1, with the lower right corner left empty. A legal move consists of sliding a block adjacent to the empty space into the empty space. Thus, from the starting placement, block 12 or 15 may be slid into the empty space. The object of the puzzle is to use a sequence of legal moves to switch the positions of blocks 14 and 15 while returning all other blocks to their original positions.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 1. The starting position for the 15-puzzle. The shaded square is left empty.

Loyd writes of how he "drove the entire world crazy," and that "A prize of $1,000, offered for the first correct solution to the problem, has never been claimed, although there are thousands of persons who say they performed the required feat." He continues,

> People became infatuated with the puzzle and ludicrous tales are told of shopkeepers who neglected to open their stores; of a distinguished clergyman who stood under a street lamp all through a wintry night trying to recall the way he had performed the feat.... Pilots are said to have wrecked their ships, and engineers rush their trains past stations. A famous Baltimore editor tells how he went for his noon lunch and was discovered by his frantic staff long past midnight pushing little pieces of pie around on a plate! [9]

The reason for this hysteria, of course, is that Loyd's puzzle has no solution. Each move causes a transposition of the 16 blocks (where the empty square is considered to contain a blank block), and for the blank to end up in the lower right corner requires an even number of moves, so the resulting permutation is even. But the desired end placement is an odd permutation of the original, and is hence unobtainable. One must assume Sam Loyd knew this, and from there one can only conjecture how much amusement he derived from driving the American public insane.

The puzzle has inspired a sizable number of articles and references in the mathematical literature. The first of these is a pair of articles published in the *American Journal of Mathematics* in 1879 by W. W. Johnson **[7]** and W. E. Story **[13]**. Johnson's article is an explanation of why odd permutations of the puzzle are impossible to obtain, while Story's article proves that all even permutations are possible. The editors were apparently so apprehensive and defensive about publishing articles on what some might charge to be a frivolous topic that they attached the following justification to the end of Story's article:

> The "15" puzzle for the last few weeks has been prominently before the American public, and may safely be said to have engaged the attention of nine out of ten persons of both sexes and of all ages and conditions of the community. But this would not have weighed with the editors to induce them to insert articles upon such a subject in the American Journal of Mathematics, but for the fact that the principle of the game has its root in what all mathematicians of the present day are aware constitutes the most subtle and characteristic conception of modern algebra, viz: the law of dichotomy applicable to the separation of the terms of every complete system of permutations into two natural and indefeasible groups, a law of the inner world of thought, which may be said to prefigure the polar relation of left and right-handed screws, or of objects in space and their reflexions in a mirror. Accordingly the editors have thought that they would be doing no disservice to their science, but rather promoting its interests by exhibiting this *à priori* polar law under a concrete form, through the medium of a game which has taken so strong a hold upon the thought of the country that it may almost be said to have risen to the importance of a national institution. Whoever has made himself master of it may fairly be said to have taken his first lesson in the theory of determinants. **[13**, p. 404]

The puzzle is a popular topic for books on recreational mathematics or mathematical potpourri, such as **[1]**, [2], **[4]**, **[5]**, **[9]**, and **[12]**, most of which use it as an example to illustrate the consequences of even and odd permutations, as does **[14]**. Various sources have suggested variants of the 15-puzzle, including **[3]**, **[4]**, **[6]**, **[8]**, **[10]**, and **[15]**. Today the puzzle appears on some computer screen savers, and a version is distributed with every Macintosh computer.

Most references to the 15-puzzle explain the impossibility of obtaining odd permutations and many state Story's result that every even permutation is indeed possible, but this author found only three proofs. R. M. Wilson **[15]** published a more general result in 1974, which we discuss at the end of this article. Ball and Coxeter's book **[1]** refers to **[10]** for a proof, but the article does not fulfill the promise. The arcane terminology of Story's article **[13]** renders it difficult to wade through, and of course it does not take advantage of modern notation developed since then. Spitznagel **[11]** published a proof in 1967, but later wrote that "Over the years there have been published a number of unnecessarily complicated explanations of the puzzle. I confess that I myself once published one of these overly complicated accounts" **[12]**. Indeed, Herstein and Kaplansky **[5]** write that

"No really easy proof seems to be known." This article intends to that deficiency.

2. SOLUTION. It should be noted that the proof provided here was developed independently of the previous proofs, but coincidentally shares some ideas with Story's proof **[13]**.

We call each of the 15 pieces *blocks*, and the 16 different squares on the board we call *cells*. For reasons that soon become apparent, we number the cells in the snakelike pattern shown in Figure 2. We can think of the empty cell as being occupied by a *blank block*. Each legal move then consists of "moving the blank," that is, exchanging the blank block with one of its horizontal or vertical neighbors. A *placement* is a bijection from the set of blocks (including the blank) to the set of cells—in other words, a snapshot of the board between moves. Given an initial placement, we wish to determine what other placements are attainable through a sequence of legal moves.

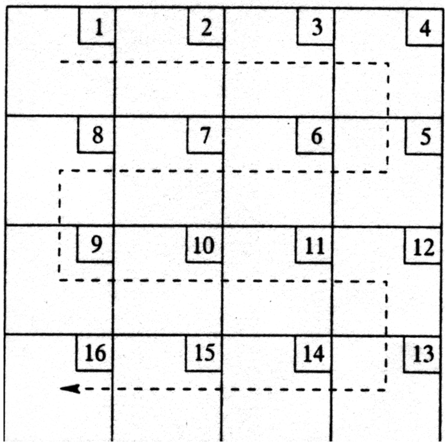

Figure 2. The dashed line and the numbers in the corner of each cell indicate a special ordering of the cells that we use to define equivalence classes of placements.

Notice that by moving the blank block along the snaking path of Figure 2 we can move the blank to any cell without changing the order of the remaining blocks along this path. This leads us to define an equivalence relation on the set of placements, two placements being equivalent if we can obtain one from the other by moving the blank along the snaking path. Each equivalence class is called a *configuration*, and contains 16 placements, one for each cell the blank can occupy. If block i occupies cell j and the blank occupies a higher numbered cell, then we say block i is in *slot* j; otherwise it is in slot $(j-1)$. Refer to Figure 3 for an example. All placements in a given configuration have the 15 blocks in the same slots, so we can denote a configuration by $[a_1, \dots, a_{15}]$, where a_i is the slot that block i occupies in the configuration.

Every move of the blank block effects a permutation on the slots occupied by the blocks. For example, moving the blank from cell 10 to cell 15 causes the permutation $(10, 11, 12, 13, 14)$ because the block originally in cell 15 (slot 14) is moved to cell 10 (which becomes slot 10) and the blocks in cells 11 through 14 are bumped up one slot. A configuration $[a_1, \dots, a_{15}]$ subjected to the permutation σ is transformed into the configuration $[a_1, \dots, a_{15}]\sigma = [a_1\sigma, \dots, a_{15}\sigma]$; since our permutations act on the right, we multiply them left to right. See Figure 3 for an example.

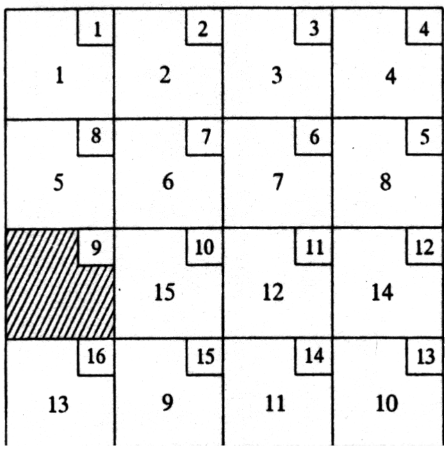

Figure 3. The placement shown here corresponds to the configuration $C = [1, 2, 3, 4, 8, 7, 6, 5, 14, 12, 13, 10, 15, 11, 9]$. Since the initial placement of Figure 1 corresponds to the configuration $I = [1, 2, 3, 4, 8, 7, 6, 5, 9, 10, 11, 12, 15, 14, 13]$, subjecting the initial configuration to the permutation $\sigma = (9, 14, 11, 13)(10, 12)$ yields C. This is an even permutation, so by Theorem 3, C is obtainable from I.

Let $\sigma_{i,j}$ denote the permutation achieved by moving the blank from cell i to cell j. Then clearly $\sigma_{i,i+1}$ is the identity, and $\sigma_{j,i} = \sigma_{i,j}^{-1}$. This leaves 9 permutations for us to work out. These are tabulated in Table 1. The key point is that one can move the blank along the snaking path of Figure 2 to any cell without changing the configuration. Therefore, the first nine permutations listed in Table 1 and their inverses may be applied *in any order*, so the problem reduces to identifying the subgroup of S_{15} (the symmetric group on the 15 slots) generated by these permutations. We prove that these permutations generate A_{15} (all even permutations).

TABLE 1. A summary of all possible permutations of slots attained by moving the blank block Moving the blank from cell i to cell j effects the permutation $\sigma_{i,j}$.

$\sigma_{1,8}$	$= (1, 2, 3, 4, 5, 6, 7)$
$\sigma_{2,7}$	$= (2, 3, 4, 5, 6)$
$\sigma_{3,6}$	$= (3, 4, 5)$
$\sigma_{5,12}$	$= (5, 6, 7, 8, 9, 10, 11)$
$\sigma_{6,11}$	$= (6, 7, 8, 9, 10)$
$\sigma_{7,10}$	$= (7, 8, 9)$
$\sigma_{9,16}$	$= (9, 10, 11, 12, 13, 14, 15)$
$\sigma_{10,15}$	$= (10, 11, 12, 13, 14)$
$\sigma_{11,14}$	$= (11, 12, 13)$
$\sigma_{n,n+1}$	$= id, n = 1, 2, \ldots, 15$
$\sigma_{i,j}$	$= \sigma_{j,i}^{-1}$ for all relevant $i > j$

Lemma 1. *For $n \geq 3$ the 3-cycles generate A_n*

Proof: By definition, all elements of A_n can be written as a product of an even number of transpositions. If a, b, c, and d are distinct, then $(a, b)(c, d) = (a, b, c)(a, d, c)$, $(a, b)(b, c) = (a, c, b)$, and $(a, b)(a, b) = id$. ■

For $n \geq 5$, Lemma 1 also follows directly from the fact that A_n is simple, since the set of 3-cycles is closed under conjugation. Let us call a 3-cycle *consecutive* if it is of the form $(k, k + 1, k + 2)$.

Lemma 2. *For $n \geq 3$, the consecutive 3-cycles $\{(1,2,3),(2,3,4), \quad ,(n-2, n-1, n)\}$ generate A_n.*

Proof: Since the 3-cycles generate A_n, it suffices to show that the consecutive 3-cycles generate all 3-cycles. This is trivial for $n = 3$. For $n \geq 4$ we see by induction that we can generate all 3-cycles not containing both 1 and n. To generate $(1, x, n)$, let $y \in \{1, \dots, n\} \setminus \{1, x, n\}$. Then $(1, x, n) = (y, x, n)(1, x, y)$. Of course, $(1, n, x) = (1, x, n)^2$. ■

Theorem 3. *The cycles listed in Table generate A_{15}*

Proof: Since all the cycles are odd, they are even permutations, so they generate a subgroup of A_{15}. Note that for any permutation σ we have $\sigma^{-1}(a_1, \dots, a_k)\sigma = (a_1\sigma, \dots, a_k\sigma)$. Thus,

$$(1,2,\dots,7)^{-n}(3,4,5)(1,2,\dots,7)^n \text{ yields } (1,2,3),\dots,(5,6,7);$$

$$(5,6,\dots,11)^{-n}(7,8,9)(5,6,\dots,11)^n \text{ yields } (5,6,7),\dots,(9,10,11) \text{ and}$$

$$(9,10,\dots,15)^{-n}(11,12,13)(9,10,\dots,15)^n \text{ yields } (9,10,11),\dots,(13,14,15)$$

as n assumes the values $-2, -1, 0, 1$, and 2. This constitutes all consecutive 3-cycles in S_{15}, so by Lemma 2 it generates A_{15}. ■

Thus, given any two placements Pl_1 and Pl_2 belonging to configurations Cf_1 and Cf_2, respectively, Pl_2 is obtainable from Pl_1 if and only if Cf_2 is an even permutation of Cf_1. Stated directly in terms of the placements, we see that if Pl_1 and Pl_2 have the blank in the same cell then Pl_2 is obtainable from Pl_1 if and only if Pl_2 is an even permutation of the 15 numbered blocks in Pl_1. Let n be the number of moves the blank cell in Pl_1 is away from the blank cell in Pl_2. Since each move of the blank block causes a transposition of two blocks, then for n odd (respectively even) Pl_2 is obtainable from Pl_1 if and only if Pl_2 is an odd (respectively even) permutation of the 16 blocks in Pl_1.

3. GENERALIZATIONS. What follows is, in some sense, the broadest generalization of the 15-puzzle. Given any connected graph on n vertices, we can label the vertices with n labels, one of which we call the *blank label*. Each move consists of interchanging the blank label with the label on an adjacent vertex. We then ask which of the $n!$ labelings may be obtained from a given initial labeling through a sequence of moves. More precisely, we ask what permutations of the $(n-1)$ ordinary labels (a subgroup of S_{n-1}) can be obtained by a sequence of moves that returns the blank to its original vertex v (since the subgroups obtained for different choices of v are isomorphic). The 15-puzzle is a special instance of this, corresponding to the graph $P_4 \times P_4$ (the cartesian product of the path on four vertices with itself) depicted in Figure 4. The vertices correspond to cells, the labels (not depicted) correspond to blocks, and the edges show which cells are adjacent.

The crux of the method presented in Section 2 lies in inducing equivalence classes and defining slots by the position of the blank along a hamiltonian path (a path that visits every vertex of the graph exactly once). The method is applicable to any graph containing a hamiltonian path, using any such path. Thus, for the 15-puzzle we could have used a spiral instead of the serpentine pattern of Figure 2. Another example is the Petersen graph. Numbering the vertices as in Figure 5, we

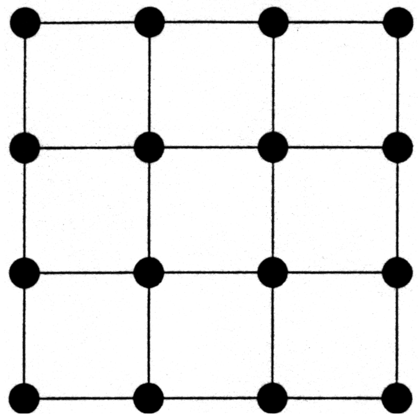

Figure 4. The graph $P_4 \times P_4$.

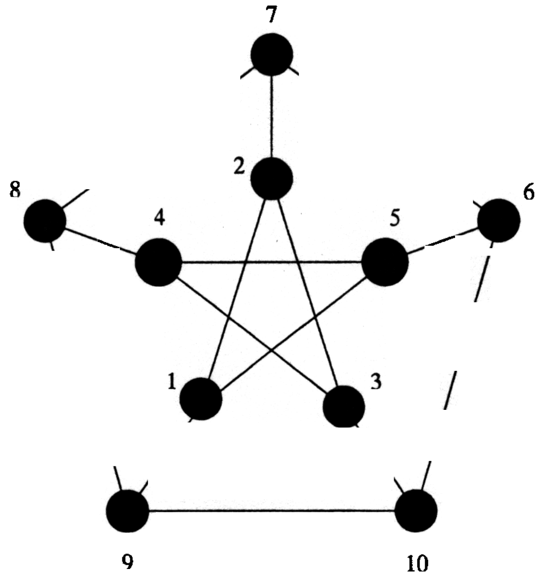

Figure 5. For the famous Petersen graph, each labeling is obtainable from every other by a sequence of legal moves. The vertices are numbered to indicate a hamiltonian path.

see that our desired group is generated by $\sigma_{1,9}$, $\sigma_{1,5}$, $\sigma_{2,7}$, $\sigma_{3,10}$, $\sigma_{4,8}$, and $\sigma_{6,10}$, where $\sigma_{i,j} = (i, i+1, \dots, j-1)$ is the permutation of slots effected by moving the blank label from vertex i to vertex j. Some calculation shows that the group generated is all of S_9; **[15]** explains why this is no coincidence.

We now discuss the general case, where the graph may or may not contain a hamiltonian path. If the graph contains a cut vertex v then none of the labels other than the blank may be moved across v, so the problem decomposes into two parts. Thus, it suffices to consider graphs containing no cut vertices.

In **[15]**, R. M. Wilson solves this case completely. Wilson's amazing result is that with the exception of cycles C_n and the graph θ_0 depicted in Figure 6, the group contains A_{n-1}. Clearly the group contains an odd permutation if and only if the graph contains an odd cycle, that is, the graph is not bipartite. So for bipartite graphs the group is exactly A_{n-1}, and otherwise it is all of S_{n-1}. Thus, aside from the two exceptional cases, either exactly *half* or *all* of the $n!$ labelings are obtainable, depending on whether or not the graph is bipartite. For θ_0, the desired

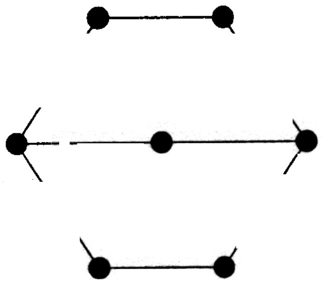

Figure 6. The graph θ_0.

group is $PGL_2(\mathbb{Z}/5\mathbb{Z})$ acting on the projective line over $\mathbb{Z}/5\mathbb{Z}$ (a group of order 120 acting 3-transitively on a set of six elements), yielding six inequivalent labelings. For C_n, the group is $\langle(1, 2, \ldots, n-1)\rangle$, yielding $(n-2)!$ inequivalent labelings. The existence of such a simple complete characterization is surprising. However, Wilson's proof, while elegant, requires considerably more sophisticated mathematics than the simple and elementary proof provided here for the special case of the 15-puzzle.

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REFERENCES

1. W. W. R. Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 12th ed., U. of Toronto Press, Toronto and Buffalo, 1974, pp. 313–316.
2. J. D. Beasley, *The Mathematics of Games*, Oxford U. Press, Oxford and New York, 1989, pp. 80–81.
3. A. L. Davies, Rotating the fifteen puzzle, *Math. Gazette* **54** (1970) 237–240.
4. M. Gardner, *Martin Gardner's Sixth Book of Mathematical Diversions from Scientific American*, U. of Chicago Press, Chicago, 1971, pp. 64–70.
5. I. N. Herstein and I. Kaplansky, *Matters Mathematical*, Chelsea, New York, 1978, pp. 114–115.
6. S. Hurd and D. Trautman, The knight's tour on the 15-puzzle, *Math. Mag.* **66** (1993) 159–166.
7. W. W. Johnson, Note on the "15" puzzle, *Amer. J. Math.* **2** (1879) 397–399.
8. H. Liebeck, Some generalizations of the 14–15 puzzle, *Math. Mag.* **44** (1971) 185–189.
9. S. Loyd, *Mathematical Puzzles of Sam Loyd: Selected and Edited by Martin Gardner*, Dover, New York, 1959, pp. 19–20.
10. H. V. Mallison, An array of squares, *Math. Gazette* **24** (1940) 119–121.
11. E. L. Spitznagel, Jr., A new look at the fifteen puzzle, *Math. Mag.* **40** (1967) 171–174.
12. E. L. Spitznagel, Jr., *Selected Topics in Mathematics*, Holt, Rinehart and Winston, New York, 1971, pp. 143–148.
13. W. E. Story, Note on the "15" puzzle, *Amer. J. Math.* **2** (1879) 399–404.
14. F. J. W. Whipple, The sign of a term in the expansion of a determinant, *Math. Gazette* **13** (1926) 126.
15. R. M. Wilson, Graph puzzles, homotopy, and the alternating group, *J. Combin. Theory* (Series B) **16** (1974) 86–96.

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