

1. (Census-Taker Problem) A census-taker knocks on a door and asks the woman inside how many children she has and how old they are.
“I have three daughters, their ages are whole numbers, and the product of the ages is 36,” says the mother.
“That’s not enough information,” responds the census-taker.
“I’d tell you the sum of their ages, but you’d still be stumped.”
“I wish you’d tell me something more.”
“Okay, my oldest daughter Annie likes dogs.”
What are the ages of the three daughters?

2. Karl and Barbara invite 10 couples to a party at their house. Karl asks everyone present, including Barbara, how many people they shook hands with. It turns out that everyone shook hands with a different number of people. If we assume that no one shook hands with his or her partner, how many people did Barbara shake hands with?

3. There is a basket of eggs.
If you remove the eggs two at a time, you wind up with one egg remaining.
If you remove them three at a time, then two wind up left in the basket.
Four at a time leaves three, five at a time leaves four, and six at a time leaves five.
If you remove seven eggs at a time, no eggs remain. None.

What is the least number of eggs that could be in the basket?

4. Alice, Bob, and Charlie each have either a red hat or a blue hat on their head. The hats were placed randomly (one’s hat color has no effect on the other) and no person knows the color of his or her hat, but each can see the other two.

Once the hats are placed, no communication of any sort is allowed and once they have all seen the others’ hats, they must simultaneously guess the color of their own hat or pass. The three will share a large monetary prize if at least one of them guesses correctly and none guesses incorrectly.

For example, they could decide that Alice will say RED and the others will PASS. This will yield the money half the time. Devise a strategy that will do better.

5. Your friend, Doctor E., is thinking of a secret number. Your job is to figure it out. To help you, here are ten statements:

- (0) The number contains exactly 2 different digit values.
- (1) No 2 digits differ by 4, 5, or 6.
- (2) More than 1 digit is a square.
- (3) Exactly 1 digit value appears more than once.
- (4) All digits are even.
- (5) The number is a palindrome.
- (6) No prime digit divides a different-valued digit.
- (7) No digit value appears more than twice.
- (8) The number contains all digit values between the largest and smallest.
- (9) All adjacent digits differ by 1.

Perhaps I should mention that I never said these are all true statements. While some are true, some are also false. As luck would have it, if a statement is true then its index number (the number to its left) appears among Doctor E's secret number's digits, and if a statement is false then its index number does not so appear. For example, if statement (8) is true, then Doctor E's number has at least one 8 in it. If statement (8) is false, then Doctor E's number has no 8s in it. Good luck!

- 6. Suppose a and b are non-zero real numbers with $a + b = 1/a + 1/b$. Show that $a^2 + b^2 = 1/a^2 + 1/b^2$.
- 7. Show that for all positive integers n , the number $3^{n+2} + 4^{2n+1}$ is divisible by 13.
- 8. What is the 2000th digit in the sequence 12345678910111213...?
- 9. Show that if n is odd, then $1^n + 2^n + \cdots + n^n$ is divisible by n^2 .