Calculus and Intermediate Real Analysis

Topics: Continuous functions, L'Hôpital's Rule, Derivative, Integral and the Fundamental Theorem, Intermediate Value, Extreme Value, Rolle's and Mean Value Theorems

Continuity: A real-valued function f is *continuous* at a if $\lim_{x\to a} f(x) = f(a)$. (Note this implies that $f(a)$ is defined and finite and that $\lim_{x\to a} f(x)$ exists.) Also, f is continuous if and only if for every sequence $\{x_n\}$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

Derivative: The derivative of $f : [a,b] \to \mathbf{R}$ at a point x in (a,b) is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x\to x_0} \frac{f(x_0) - f(x)}{x_0 - x}$$

provided this limit exists.

Fundamental Theorem: If $F(t)$ has a continuous derivative on the interval $[a,b]$, then

$$\int_a^b F'(t)dt = F(b) - F(a).$$

If f is a continuous function on an interval $[a,b]$, then for any x in (a,b)

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Mean Value Theorems: For each of the following, $f : [a,b] \to \mathbf{R}$ is continuous over $[a,b]$ and f is differentiable on (a,b).

- **Intermediate Value Theorem**: If y is between $f(a)$ and $f(b)$ then there is a number c between a and b such that $f(c) = y$.
- **Mean Value Theorem for Integrals**: There is a number c in (a,b) such that

$$\int_a^b f(t)dt = f(c)(b-a).$$

- **Extreme Value Theorem**: There are numbers c and d in $[a,b]$ such that $f(c) \le f(x) \le f(d)$ for all x in $[a,b]$. (f attains a global max and a global min on $[a,b]$.)
- **Rolle's Theorem**: If $f(a) = f(b)$, then there is a number c in (a,b) such that $f'(c) = 0$.
- **Mean Value Theorem**: There is a number c in (a,b) such that

$$\frac{f(a) - f(b)}{b-a} = f'(c).$$

This last theorem has several corollaries:

– If $F'(x) = 0$ over (a, b), then f is constant over the interval.
– If $f'(x) = g'(x)$ over (a, b), then there is a constant C such that $f(x) = g(x) + C$.
– If $f'(x) > 0$ for all x in (a, b) then f is an increasing function.

1. (Larson 6.2.4) Suppose $f : [0, 1] \to [0, 1]$ is continuous. Prove that there exists a number c in $[0, 1]$ such that $f(c) = c$.

2. (Larson 6.9.7) What function is defined by the equation

$$f(x) = \int_0^x f(t)dt + 1?$$

3. (UK 1995) Let a, b, c be real numbers satisfying $a < b < c$, $a + b + c = 6$ and $ab + bc + ca = 9$. Prove that $0 < a < 1 < b < 3 < c < 4$.

4. Let $f(x) = \prod_{k=0}^{n}(x + k)$. Find $f'(1)$. Don't use the product rule.

5. Let $f(x)$ be a differentiable function with satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If $f'(0) = 3$, find $f(x)$.

6. (Putnam 1994) Find all c such that the graph of the function $x^4 + 9x^3 + cx^2 + ax + b$ meets some line in four distinct points.

7. (Larson 6.3.2, Putnam 1967) Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)| \leq |\sin x|$ for all real x, prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

8. (Larson 6.6.2) Let $f : \mathbf{R} \to \mathbf{R}$ be such that for all x and y in $\mathbf{R}$, $|f(x) - f(y)| < (x - y)^2$. Prove that f is a constant.