In-Class Problems

Thinking outside of the box.

1. Pat wants to take a 1.5 meter long sword onto a train, but the conductor won't allow it as carry-on luggage. And the baggage person won't take any item whose greatest dimension exceeds 1 meter. What should Pat do?

2. What is the next letter in the sequence O, T, T, F, F, S, S, E, ...?

3. Fill in the next column of the table.

1	3	9	3	11	18	13	19	27	55	
2	6	2	7	15	8	17	24	34	29	
3	1	5	12	5	13	21	21	23	30	

4. The following sequence has become known as the John H. Conway sequence, after the celebrated Princeton mathematician. Can you determine the next term of the sequence?

$$1, 1, 1, 3, 1, 4, 1, 1, 3, 6, 1, 2, 3, 1, 4, 8, 1, 3, 3, 2, 4, 1, 6, ?$$

5. Find the next number in this sequence.

$$4, 8, 21, 52, 65, 96, \ldots$$

6. A certain primitive village has the following social rules: Whenever a husband is unfaithful to this wife, every other wife knows immediately, but not the wife who has been cheated on. The wives never talk to anyone about these matters, and neither do the husbands. As soon as a wife can determine irrefutably that her husband has cheated, she tattoos the letter "A" on his forehead before sundown of that same day.

   On a given day the mayor announces that there is at least one unfaithful husband in the village (he does *not*, however, say how many unfaithful husbands there are). If in fact there are 37 unfaithful husbands in the village, what will transpire?

   What happens if the mayor actually announces the number of unfaithful husbands?

Classic Pigeonhole Principle

1. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104. (Actually, 20 can be replaced by 19.)

2. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less that one half unit apart. What if the equilateral triangle is replaced by a square whose side has length one unit?

3. Every point on the plane is colored either red or blue. Prove that no matter how the coloring is done, there must exist two points, exactly one mile apart, which are the same color.

4. Prove that in any finite gathering of people, there are at least two people who know the same number of people at the gathering (assuming “knowing” is a mutual relationship).

More Pigeonhole Principle

1. People are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table, and this circular platform can rotate (this is commonly found in Chinese restaurants that specialize in banquets). Each person ordered a different entree, and it turns out that no one has the correct entree in front of him. Show that it is possible to rotate the platform so that at least *two* people will have the correct entree.

2. Color the plane in 3 colors. Prove that there are two points of the same color 1 unit apart.

   Color the plane in 2 colors. Prove that there will always exist an equilateral triangle with all its vertices of the same color.

3. Show that among any $n+1$ positive integers, there must be two whose difference is a multiple of n.

4. Given any set of ten natural numbers between 1 and 99 inclusive (decimal notation), prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.

5. Chose any $(n+1)$-element subset from $\{1, 2, \ldots, 2n\}$. Show that this subset must contain two integers that are relatively prime.

6. (Putnam 1994) Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.