Algebra: from mindless computations and ugly manipulations to elegant problem solving.

- Factor relentlessly.

$$\begin{aligned}
(x+y)^2 &= x^2+2xy+y^2\\
(x-y)^2 &= x^2-2xy+y^2\\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 = x^3+y^3+3xy(x+y)\\
(x-y)^3 &= x^3-3x^2y+3xy^2-y^3 = x^3-y^3-3xy(x-y)\\
x^2-y^2 &= (x-y)(x+y)\\
x^n-y^n &= (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})\\
x^n+y^n &= (x+y)(x^{n-1}-x^{n-2}y+\cdots-xy^{n-2}+y^{n-1}) \quad n \text{ odd}
\end{aligned}$$

- Look for an elegant solution.
- Exploit symmetry.
- Look for parity arguments.
- Add zero creatively. Multiply cleverly by one.
- Create and recognize perfect squares.

$$\begin{aligned}
(x+y+z)^2 &= x^2+y^2+z^2+2xy+2xz+2yz\\
(x+y+z+w)^2 &= x^2+y^2+z^2+w^2+2xy+2xz+2xw+2yz+2yw+2zw\\
x^2+ax &= x^2+ax+\frac{a^2}{4}-\frac{a^2}{4} = \left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2 \quad \textit{Completing the square}\\
(x-y)^2+4xy &= (x+y)^2
\end{aligned}$$

Division Algorithm for Polynomials: If $F(x)$ and $G(x)$ are polynomials over a field k (for example, k might be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}/p\mathbb{Z}$), there exist unique polynomials $Q(x)$ and $R(x)$ over the field k such that

$$F(x) = Q(x)G(x) + R(x),$$

where $R(x) = 0$ or deg $R(x) <$ deg $G(x)$

Factor Theorem If $F(x)$ is a polynomial over an integral domain D, an element a of D is a root of $F(x) = 0$ if and only if $x - a$ is a factor of $F(x)$

Identity Theorem Suppose that two polynomials in x over and integral domain are each of degree $\leq n$. If these polynomials have equal values for more than n distinct values of x, then the two polynomials are identical.

Application: If $x^2+ax+b = (x-r_1)(x-r_2)$, then $r_1+r_2 = -a$ and $r_1r_2 = b$. This can be generalized to higher degree polynomials.

Matrices: systems of linear equations, determinants, inverses

1. (Zeist 5.2.1) If $x + y = xy = 3$, find $x^3 + y^3$.

2. (Zeist 5.2.9) Factor $x^4 + 4$ into two polynomials with real coefficients.

3. (AIME 1983) What is the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$?

4. (Larson 1.10.4, EF 7.4, Putnam 1954) Let n be an odd integer greater than 1, and let A be an n-by-n symmetric matrix such that each row and each column of A consists of some permutation of the integers $1, \ldots, n$. Show that each one of the integers $1, \ldots, n$ must appear in the main diagonal of A.

5. (Larson 3.2.13a) Determine whether the following matrix is singular or nonsingular:
$$\begin{pmatrix} 54401 & 57668 & 15982 & 103790 \\ 33223 & 26563 & 23165 & 71489 \\ 36799 & 37189 & 16596 & 46152 \\ 21689 & 55538 & 79922 & 51237 \end{pmatrix}$$

6. (EF 7.3) Let a, b, c be odd integers. Show that the quadratic equation
$$ax^2 + bx + c = 0$$
does not have a rational solution.

7. (Larson 1.10.10, Putnam 1954) Show that $x^2 - y^2 = a^3$ always has integral solutions for x and y whenever a is a positive integer.

8. (Larson 3.3.14, Putnam 1947) Given distinct integers a, b, c, d such that
$$(x - a)(x - b)(x - c)(x - d) - 4 = 0$$
has an integral root r, show that $4r = a + b + c + d$.

9. (Larsen 4.2.3) Given the polynomial $F(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with integral coefficients $a_0, a_1, \ldots, a_{n-1}$, and given also that there exist four distinct integers a, b, c, d such that $F(a) = F(b) = F(c) = F(d) = 5$, show that there is no integer k such that $F(k) = 8$.

10. (Larson 4.3.8) Let x_1 and x_2 be the roots of the equation
$$x^2 - (a + d)x + (ad - bc) = 0.$$
Show that x_1^3 and x_2^3 are the roots of
$$y^2 - (a^3 + d^3 + 3abc + 3bcd)y + (ad - bc)^3 = 0.$$