In-Class Exercises  
October 25, 2001

Topics: Binomial Coefficients, Geometric Series, Telescoping Series, Power Series

**Binomial Coefficients:** \( \binom{n}{k} \) can be expanded in many ways:

\[
\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)}{k(k-1)} = \binom{n-1}{k-1} + \binom{n-1}{k-1}
\]

\[
(x + y)^n = \sum_k \binom{n}{k} x^k y^{n-k}
\]

\[
\binom{r+s}{n} = \sum_k \binom{r}{k} \binom{s}{n-k}
\]

\[
\binom{n+k+1}{k} = \sum_k \binom{n+k}{k}
\]

**Geometric Series:**

\[
\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \quad |x| < 1
\]

\[
\sum_{i=1}^{n} x^i = \frac{1-x^{n+1}}{1-x}, \quad x \neq 1
\]

**Partial Fractions:**

\[
\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}
\]

**Power Series:** Given a power series \( \sum_{i=0}^{\infty} a_i x^i \), exactly one of the following holds:

1. Radius of convergence \( \infty \): Series converges for all real \( x \).
2. Radius of convergence \( 0 \): Series converges iff \( x = 0 \).
3. Radius of convergence \( r \): Series converges for all real \( |x| < r \) and diverges for \( |x| > r \).

A power series can be found for \( f \) if it can be differentiated as many times as we like in an open interval \( |x-a| < r \) around \( a \), in which case, \( f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n \) for \( |x| < r \).

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

\[
sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\]

\[
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\]

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1
\]

\[
\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}
\]

\[
(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n, \quad r \text{ real and } |x| < 1
\]
1. Write

\[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100} \]

as a fraction in lowest terms.

2. Find the sum \( 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! \).

3. Find the sum \( \binom{n}{1} + 2 \binom{n}{2} + 3 \binom{n}{3} + \cdots + n \binom{n}{n} \). (Hint: use the in-and-out formula)

4. Find a formula for the sum \( \sum_{k=1}^{n} \frac{k}{(k + 1)!} \).

5. Find the sum of the infinite series \( \sum_{n=1}^{\infty} \frac{1}{2n^2 - n} \).

6. Evaluate \( f(n) = 1^2 - 2^2 + 3^2 - \cdots + (2n - 1)^2 - (2n)^2 \).

7. Simplify the product

\[
\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{a^2}\right) \left(1 + \frac{1}{a^4}\right) \cdots \left(1 + \frac{1}{a^{2^{100}}}\right)
\]

8. Evaluate \( \sum_{n=1}^{\infty} \frac{n^3}{n!} \).

9. (Larson 5.3.1) Sum the infinite series \( \sum_{i=1}^{\infty} \frac{1}{(3i - 2)(3i + 1)} \).

10. (Putnam 1977) Evaluate the infinite product \( \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} \).

11. (Putnam 1977) For \( 0 < x < 1 \), express \( \sum_{n=0}^{\infty} \frac{x^{2n}}{1 - x^{2n+r}} \) as a rational function of \( x \).

12. (Putnam 1984) Evaluate \( \sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} \) as a rational number.