MCS-200 **In-Class Exercises** October 25, 2001

Topics: Binomial Coefficients, Geometric Series, Telescoping Series, Power Series

Binomial Coefficients: $\binom{n}{k}$ can be expanded in many ways:

$$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\begin{aligned}
(x+y)^n &= \sum_k \binom{n}{k} x^k y^{n-k} \\
\binom{r+s}{n} &= \sum_k \binom{r}{k}\binom{s}{n-k} \\
\binom{n+k+1}{k} &= \sum_k \binom{n+k}{k}
\end{aligned}$$

Geometric Series:

$$\begin{aligned}
\sum_{i=0}^{\infty} x^i &= \frac{1}{1-x}, \quad |x| < 1 \\
\sum_{i=1}^{n} x^i &= \frac{1-x^{n+1}}{1-x}, \quad x \neq 1
\end{aligned}$$

Partial Fractions: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Power Series: Given a power series $\sum_{i=0}^{\infty} a_i x^i$, exactly one of the following holds:

1. Radius of convergence ∞: Series converges for all real x.
2. Radius of convergence 0: Series converges iff $x = 0$.
3. Radius of convergence r: Series converges for all real $|x| < r$ and diverges for $|x| > r$.

A power series can be found for f if it can be differentiated as many times as we like in an open interval $|x - a| < r$ around a, in which case, $f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$ for $|x| < r$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad (1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n, \quad r \text{ real and } |x| < 1$$

1. Write

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}$$

as a fraction in lowest terms.

2. Find the sum $1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!$.

3. Find the sum $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}$. (Hint: use the in-and-out formula)

4. Find a forumla for the sum $\sum_{k=1}^{n}\frac{k}{(k+1)!}$.

5. Find the sum of the infinite series $\sum_{n=1}^{\infty}\frac{1}{2n^2-n}$.

6. Evaluate $f(n)=1^2-2^2+3^2-\cdots+(2n-1)^2-(2n)^2$.

7. Simplify the product

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{a^4}\right)\cdots\left(1+\frac{1}{a^{2^{100}}}\right)$$

8. Evaluate $\sum_{n=1}^{\infty}\frac{n^3}{n!}$.

9. (Larson 5.3.1) Sum the infinite series $\sum_{i=1}^{\infty}\frac{1}{(3i-2)(3i+1)}$.

10. (Putnam 1977) Evaluate the infinite product $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$

11. (Putnam 1977) For $0<x<1$, express $\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+1}}}$ as a rational function of x.

12. (Putnam 1984) Evaluate $\sum_{k=1}^{\infty}\frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}$ as a rational number.