Team Practice: Seventh Annual Iowa Collegiate Mathematics Competition, March 31, 2001

1. **Side of a triangle.**

   If the area of an equilateral triangle is $\frac{3}{4}$, what is the length of each side?

2. **Solve for x.**

   Find all real solutions of the equation $\lfloor x \rfloor^2 - 5\lfloor x \rfloor - 6 = 0$. Here $\lfloor x \rfloor$ denotes, as usual, the greatest integer less than or equal to x.

3. **Taylor's Theorem.**

   According to Taylor's Theorem, if f is a twice differentiable function on an interval containing a and b, with $a \neq b$, then there is a number c between a and b such that

   $$f(b) - f(a) = f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2.$$

   Express c as simply as possible in terms of a and b if $f(x) = x^3$.

4. **An irrational number.**

   Let r and s be positive rational numbers with $\sqrt{r}$ irrational. Prove that $\sqrt{r}+\sqrt{s}$ is irrational.

5. **Multiplicative inverses.**

   Let R be the ring of integers modulo 2001. For example, in R, $1900+125 = 24$ and $90^2 = 96$.

   (a) Determine whether the element 1334 has a multiplicative inverse in R, and if so, find it. If not, show this.

   (b) Do the same for the element 1333.

6. **A harmonic identity.**

   For each positive integer n, let $h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

   Prove that for every integer $n \geq 2$, $n + h(1) + h(2) + \cdots + h(n-1) = nh(n)$.

7. **Sum of squares divisible by n.**

   A certain set of n integers has the property that the difference between the product of any $n-1$ of them and the remaining one is divisible by n. Prove that the sum of the squares of all n integers is divisible by n.

8. **Sum the series.**

   Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{3^n}$, and justify your answer.

9. **A multiple of 49.**

   After several applications of the operation of differentiation and the operation of multiplication by $x-1$, performed in unspecified order, the polynomial $x^8 + x^7$ is changed to $ax+b$, where $a \neq 0$.

   Prove that $a-b$ is an integer divisible by 49.

10. **Integer roots.**

    Find all real numbers p such that all three roots of the cubic equation $5x^3 - 5(p+1)x^2 + (71p-1)x + 1 = 66p$ are positive integers.