Inequalities

Basic Arithmetic:

- *Addition.* If $x \geq y$ and $a \geq b$, then $x + a \geq y + b$.
- *Multiplication.* If $x \geq y$ and $a \geq 0$, then $ax \geq by$. Conversely, if $a < 0$, then $ax \leq ay$.
- *Reciprocals.* If $x \geq y$, then $1/x \leq 1/y$, provided that both x and y have the same sign.
- *Distance interpretation of the absolute value.* The set

$$\{x \text{ such that } |x - a| = b\}$$

consists of all points x on the real number line that lie within a distance b of the point a.

AM-GM: Arithmetic-mean - geometric-mean inequality

$$\frac{a+b}{2} \geq \sqrt{ab}, \quad 0 < a \leq b,$$

or more generally

$$\frac{a_1 + a_2 + \cdots a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}$$

with equality if and only if all the a_i's are equal.

Another inequality that comes up frequently is:

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

Cauchy-Schwarz: If $a_i > 0$ and $b_i > 0$ are defined for $1 \leq i \leq n$,

$$\sum_i a_i b_i \leq \sqrt{\left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)}$$

with equality if and only if all the ratios a_i/b_i are equal.

Derivatives: Derivatives can be useful. For example, if $f(0) = 0$ and $f'(x) > 0$ for $x > 0$, then we know $f(x) > 0$ for $x > 0$.

Integrals: show series is LHS or RHS for monotonic function

Series: To compare $f(x)$ and $g(x)$ compare their Taylor series expansions.

1. Which is larger, $\sqrt{19} + \sqrt{99}$ or $\sqrt{20} + \sqrt{98}$?

2. Which is larger $\frac{1998}{1999}$ or $\frac{1999}{2000}$?

3. Which is bigger 2000! or 1000^{2000}?

4. Prove that $\dfrac{(x+y+z)^2}{3} \leq x^2 + y^2 + z^2$.

5. (a) Find the maximum value of $x^{1/x}$ for $x > 0$. Without doing any numerical calculations, decide which is bigger, π^e or e^π.

   (b) (UIUC, 1997-2.1) Determine which of the two expressions $\sqrt{n}^{\sqrt{n+1}}$ and $\sqrt{n+1}^{\sqrt{n}}$ is larger when n is an integer greater than 8.

6. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers. Show that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$$

   with equality holding if and only if the a_i are equal.

7. (Larsen 7.2.3)

   If a, b, c are positive numbers such that $(1+a)(1+b)(1+c) = 8$, prove that $abc \leq 1$.

8. (Larsen 7.3.3)

   Given that a, b, c, d, e are real numbers such that

$$\begin{aligned} a + b + c + d + e &= 8, \\ a^2 + b^2 + c^2 + d^2 + e^2 &= 16, \end{aligned}$$

   determine the maximum value of e.

9. (a) (UIUC, 1999 2.1) Let $x_1, x_2, \ldots, x_n$ be the numbers $1, 2, \ldots, n$, written in some order. Prove that

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \leq 1^2 + 2^2 + \cdots + n^2.$$

   (b) (UIUC, 1997-3.4) Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let $b_1, b_2, \ldots, b_n$ be a permutation of the a_i's. Show that $\displaystyle\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n$.

   (c) (Larsen 7.3.6) Use the Cauchy-Schwarz inequality to prove that if $a_1 + \cdots + a_n = 1$, then $a_1^2 + \cdots a_n^2 \geq 1/n$.

   (d) (UIUC, 1997 4.3) Let $a_i, a_2, \ldots, a_n$ be real numbers with $\displaystyle\sum_{i=1}^{n} a_i = 1$. Prove that

$$\sum_{i=1}^{n} i a_i^2 > \frac{1}{2\sqrt{n}}.$$