Inequalities

Basic Arithmetic:

- **Addition.** If $x \geq y$ and $a \geq b$, then $x + a \geq y + b$.
- **Multiplication.** If $x \geq y$ and $a \geq 0$, then $ax \geq by$. Conversely, if $a < 0$, then $ax \leq ay$.
- **Reciprocals.** If $x \geq y$, then $1/x \leq 1/y$, provided that both $x$ and $y$ have the same sign.
- **Distance interpretation of the absolute value.** The set

$$\{x \text{ such that } |x - a| = b\}$$

consists of all points $x$ on the real number line that lie within a distance $b$ of the point $a$.

**AM-GM:** Arithmetic-mean - geometric-mean inequality

$$\frac{a + b}{2} \geq \sqrt{ab}, \quad 0 < a \leq b,$$

or more generally

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}$$

with equality if and only if all the $a_i$’s are equal.

Another inequality that comes up frequently is:

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

**Cauchy-Schwarz:** If $a_i > 0$ and $b_i > 0$ are defined for $1 \leq i \leq n$,

$$\sum_i a_i b_i \leq \sqrt{\left(\sum_i a_i^2\right) \left(\sum_i b_i^2\right)}$$

with equality if and only if all the ratios $a_i/b_i$ are equal.

**Derivatives:** Derivatives can be useful. For example, if $f(0) = 0$ and $f'(x) > 0$ for $x > 0$, then we know $f(x) > 0$ for $x > 0$.

**Integrals:** show series is LHS or RHS for monotonic function

**Series:** To compare $f(x)$ and $g(x)$ compare their Taylor series expansions.
1. Which is larger, $\sqrt{19} + \sqrt{99}$ or $\sqrt{20} + \sqrt{98}$?

2. Which is larger $\frac{1998}{1999}$ or $\frac{1999}{2000}$?

3. Which is bigger $2000!$ or $1000!$?

4. Prove that $\frac{(x + y + z)^2}{3} \leq x^2 + y^2 + z^2$.

5. (a) Find the maximum value of $x^{1/x}$ for $x > 0$. Without doing any numerical calculations, decide which is bigger, $\pi^e$ or $e^\pi$.

   (b) (UIUC, 1997-2.1) Determine which of the two expressions $\sqrt{n^{n+1}}$ and $\sqrt{n+1}^{\sqrt{n}}$ is larger when $n$ is an integer greater than 8.

6. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers. Show that

   \[
   (a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq n^2
   \]

   with equality holding if and only if the $a_i$ are equal.

7. (Larsen 7.2.3)

   If $a, b, c$ are positive numbers such that $(1 + a)(1 + b)(1 + c) = 8$, prove that $abc \leq 1$.

8. (Larsen 7.3.3)

   Given that $a, b, c, d, e$ are real numbers such that

   \[
   a + b + c + d + e = 8, \\
   a^2 + b^2 + c^2 + d^2 + e^2 = 16,
   \]

   determine the maximum value of $e$.

9. (a) (UIUC, 1999 2.1) Let $x_1, x_2, \ldots, x_n$ be the numbers $1, 2, \ldots, n$, written in some order. Prove that

   \[
   x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1 \leq 1^2 + 2^2 + \cdots + n^2.
   \]

   (b) (UIUC, 1997-3.4) Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let $b_1, b_2, \ldots, b_n$ be a permutation of the $a_i$’s. Show that $\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n$.

   (c) (Larsen 7.3.6) Use the Cauchy-Schwarz inequality to prove that if $a_1 + \cdots + a_n = 1$, then $a_1^2 + \cdots + a_n^2 \geq 1/n$.

   (d) (UIUC, 1997 4.3) Let $a_1, a_2, \ldots, a_n$ be real numbers with $\sum_{i=1}^{n} a_i = 1$. Prove that

   \[
   \sum_{i=1}^{n} i a_i^2 > \frac{1}{2\sqrt{n}}.
   \]