Abstract

This article describes the collaboration of three recent Ph.Ds, each teaching abstract algebra for the first time. The authors describe how they met, how they collaborated and how they addressed various pedagogical issues. In particular they examine issues that relate to upper division proof writing courses. All three of the authors emphasized and encouraged active student participation in their courses. This paper provides a discussion of techniques used and examples of assignments used to encourage active learning at the beginning of the course, in the classroom, outside of the classroom and through assessment. Included are notes given to students on the syllabus, some sample exploratory group activities, techniques to encourage conjecture-making and proof-building, methods for refining proof-writing, and various ways of structuring homework.
Active Learning in Abstract Algebra: An Arsenal of Techniques
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AUTHOR DESCRIPTIONS:
Dr. Laurie Burton is an Assistant Professor of Mathematics at Western Oregon University where she transferred after spending four years at Central Washington University. She is trained as a commutative ring theorist and is interested in mathematics education. She enjoys designing classes that utilize different educational approaches to enhance active student learning. After attending the California State University at Chico as an undergraduate, she studied at the University of Oregon for her Ph.D., which she received in 1995.

Dr. Sarah-Marie Belcastro is an Assistant Professor of Mathematics at the University of Northern Iowa. She specializes in algebraic geometry (surfaces which are hypersurfaces in toric varieties) and the mathematics of paper folding. After attending Haverford College as an undergraduate, she studied at the University of Michigan for her Ph.D., which she received in 1997. In addition to being a mathematician, Sarah-Marie is also a member of a dance company and thinks about feminist philosophy of science.

Dr. Moira McDermott is an Assistant Professor of Mathematics at Gustavus Adolphus College. She specializes in commutative algebra and is particularly interested in characteristic p methods and computational algebra. After attending Bryn Mawr College as an undergraduate, she studied at the University of Michigan for her Ph.D., which she received in 1996. She was a Visiting Assistant Professor at Bowdoin College for two years before joining the faculty of Gustavus.

BACKGROUND AND PURPOSE:
Laurie, Sarah-Marie and Moira met as fellows of the 1997-1998 Project NExT* cohort. We discovered through email conversations preceding our physical meeting in Atlanta, Georgia (Mathfest, 1997) that we were all scheduled to teach abstract algebra courses the following academic year. When we met in Atlanta we discussed various text possibilities, our plans for active presentation of course content and the structure of our courses. As a collective we decided to use Joe Gallian’s text, Contemporary Abstract Algebra ([1]) and to communicate via email about our day to day experiences and work with our students. We then communicated throughout the year and we collaborated again, in person, at the AMS-MAA 1998 Joint Mathematics Meetings in Baltimore and at the 1998 Mathfest in Toronto. As professors in our first years of teaching we were all excited about teaching abstract algebra, and we were particularly interested in teaching our courses with an emphasis on active student participation. We shared the opinion that requiring active preparation and active involvement of ourselves would result in a greater commitment to our courses from our students. This article summarizes our discussions, collaboration and findings, explaining and giving examples of some of our techniques as outlined below.

* Project NExT (New Experiences in Teaching), funded by the MAA and the Exxon Education Foundation, is a program for new or recent Ph.D.s in the mathematical sciences who are interested in improving the teaching and learning of undergraduate mathematics. Faculty who are just beginning or just completing their first year of full-time teaching at the college/university level are invited to apply to become Project NExT Fellows. For more information please see the MAA webpage at: http://archives.math.utk.edu/projnexit/
OUTLINE

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OUR COURSES AND OUR STUDENTS

Laurie taught two different abstract algebra courses, a summer Master’s in Teaching class and a year-long senior level undergraduate sequence. The graduate students were highly motivated to succeed but were not uniformly strong in their undergraduate preparation. These students needed to increase their mathematical maturity and to be given materials they could use in their high school classrooms. In the senior sequence the undergraduates were still struggling with basic logic and the idea of a proof. These students needed to learn to work with higher level mathematics.

sarah-marie taught a year-long sequence, mainly for secondary mathematics education majors. Her students varied widely in their motivation and preparation, but most were somewhat weak. In particular, many knew nothing about the rudiments of proof, were passive learners, and expected that they would not have to spend large amounts of time on the class.

Moira taught a two semester sequence aimed primarily at math majors. Her students were generally strong although several had little or no experience writing proofs.

WAYS TO START STUDENTS ON ACTIVE LEARNING

As with all courses, it is important to set an enthusiastic yet hard-working tone in the classroom from the beginning of the term. We each provided our students with a syllabus which clearly outlined our expectations. To help the students realize active proof writing would be an important focus of the course, Laurie included the following on her undergraduate course syllabus:

Example: Notes on Proof Writing
Tests are fun, but the real focus of this course will be on successfully writing the homework.

Please keep in mind when you write up your homework that you should:

- Clearly (re)state the problem.
- Clearly state any assumptions you are making.
- Clearly state which proof method you are using if other than a straightforward proof (e.g. "I will proceed using proof by contrapositive").
- Carefully show each step you are taking.
- Clearly reference any results that you are using.
- Conclude your proof.

Please also keep the following in mind when you read over your homework before turning it in:

- Are the spelling, grammar, and punctuation correct?
- Is the mathematics correct?
- Did you answer the question or prove the statement that was originally asked or given?
- Is this paper neat, organized, and pleasingly presented?

And last but definitely not least:

READ over each sentence of your proof. Try reading your proof out loud. Does it make sense? Is each sentence complete? Are any steps left out (any amazing and unverified leaps?)?

sarah-marie was not prepared for the passivity of her students, so she did not explicitly address the issue of active learning in her initial handouts. Over the course of her first term she realized she needed to remind her students frequently that it is normal to read the book over and over. She also found it necessary to explicitly state that she expected them to spend at least two hours per week reading the book and at least eight hours per week on homework. The second term, she added the following to her first day handout:

**Example: Notes on Style**

You can think of this as a seminar course if you would like a pigeonhole to put it in. There will be at most one third of each class spent on lecturing, and in reality, the average amount of lecture for the first half of the course will be about five minutes per class. Most of class will be spent answering your questions and working on problems. I view classroom time not as a time when you do most of your learning, but as a focus time where you get unstuck, and/or a boost, and/or a burst of insight. Most of your learning will take place outside the classroom, on your own time.

The continuing students helped reinforce this paradigm for the newer students.

In order to emphasize the point that class would not be just lectures, Moira had her students work on a project on the first day of class.

**Example: First Day Activity**

Students were given cardboard triangles and squares with numbered corners. They were first asked to determine all of the symmetries of the figures. They
were given instructions on how to do this based on the information about sym-
metries in Chapter 1 of [1]. They then came up with labels for the symmetries,
for example, “rotation by 90 degrees”. Moira drew a grid on the board and
had each student fill in one column of the multiplication table. The students
were then asked to come up with interesting properties of the table. This led
to a discussion that introduced the ideas of closure, commutativity, identities
and inverses.

This activity served two purposes. First, it got students doing something
other than sitting passively on the first day. Second, it previewed basic ideas
that came up in the beginning of the course and gave students a concrete
example to which they could refer as the course progressed.

ENCOURAGING PARTICIPATION: ACTIVE STUDENTS INSIDE THE
CLASSROOM

Of course just suggesting to students that they need to actively participate in the course
and write coherent proofs does not create a dynamic classroom and skilled proof writers.

We all regularly had our students do group work in class. Group activities varied from
working on selected text problems, exploratory exercises and peer analysis of proof to
completing instructor-designed worksheets. By facilitating group activities we encouraged
our students to explore the current material and helped them to increase their proof writing
skills. It was clear the intensive interaction with and between our students in the classroom
added to the overall successes of our courses. We found many students who are not willing
to speak up in a large classroom are willing to speak up in small groups. We also noted the
comfortable environment of group work helped some students feel less intimidated about
seeking help during office hours. In fact, throughout our courses, we all saw students who
worked together in class getting to know each other. This helped our students to form
valuable study groups outside of the classroom.

The mechanics of managing collaborative group activities in an abstract algebra class are
almost identical to those for a calculus class. We recommend [2], [3] and [4] to the interested
reader.

Here are some examples of techniques we used to promote participation in the classroom.

Example: Avoiding Lecturing

In order to help the students successfully learn to construct proofs, sarah-
marie forced the students to read the text carefully. She often began class
by asking students whether they had gotten stuck in the reading, and if so,
where. Upon receiving the expected silence, she would ask the students to
break into groups (average size 4) and work on two or three very simple
problems. Invariably, the students were quickly stuck. sarah-marie would
move from group to group, asking for definitions or statements of related
theorems. As the course progressed, this situation changed from sarah-marie
repeatedly reminding the students to repeatedly look up definitions, to the
students automatically analyzing the details of definitions in order to begin
short proofs. After (and only after) students had produced and discussed
relevant definitions would sarah-marie provide assistance in the stickier points of the problems.

Examples of this sort of exercise are
- “Prove that $S_4$ is not isomorphic to $D_{12}$” (from the “Isomorphisms” chapter)
- “If $A$ and $B$ are ideals of a ring, show that the sum of $A$ and $B$, $A + B = \{a + b | a \in A, b \in B\}$, is an ideal” (from the “Ideals and Factor Rings” chapter).

These problems force the students to analyze the details of a definition or several definitions; here they must use the conditions given in the definitions of “isomorphism” and “ideal” in order to complete the proof. The students must also review the structural facts they have learned about groups like $S_4$ and $D_{12}$, which then forces them to recall concepts such as “Abelian.”

This technique was very successful in impressing on the students that proofs don’t materialize out of thin air. They learned that they needed to remind themselves of details constantly, and that it often takes a very long time to produce a proof of even a simple exercise.

Example: Exploratory Exercises

When Moira lectured she wasn’t confident her students would always ask questions about the concepts they didn’t understand. Further, she worried that they might not have the skills or be confident enough to make conjectures. To address this Moira wrote exploratory exercises for in-class group work. For example, she had her students construct groups of small order. Students were led through a series of steps that allowed them to construct the group of order three and to conclude that there is only one up to isomorphism. They also constructed two groups of order four and concluded that those were the only two. Another exercise involved determining when $U(n)$, the group of units of $\mathbb{Z}_n$, is cyclic. Students determined whether $U(n)$ was cyclic for small values of $n$. They were then given the information for larger values of $n$ and asked to form a conjecture about the prime power decomposition of integers $n$ for which $U(n)$ is cyclic. The students were allowed to ask Moira for further data and, in fact, they were able to prove their conjectures at a later point in the course. The design of this exploratory exercise was motivated by one of the Programming Exercises in the text (Chapter 4, Programming Exercise #1, [1]). In general the programming exercises in [1] are a good source of ideas for developing one’s own exploratory exercises. One way for students to get data without actually doing the programming is to use the online applets found through Joe Gallian’s webpage at the University of Minnesota, Duluth (http://www.d.umn.edu/~jgallian/msproject/project_head.html). Students liked the exploratory exercises; for many of them these exercises provided their first exposures to making conjectures.

Example: In-Class Analysis of Proofs
Laurie and Moira regularly had their undergraduates write proofs on the board. The students found discussing the material with their peers to be helpful and enjoyable. We found when students had to explain to their peers precisely what they were trying to communicate, they were better able to focus on the logical and composition flaws in their proofs. Furthermore, in the long run, this activity greatly helped the students become engaged in learning the material in the course and helped to build the students' proof writing skills and style.

We also found occasionally allowing the students to write their proofs on the board for peer analysis (before the homework was due) positively affected not only student attitude and performance but also their involvement in the course.

Another technique that Moira tried was to lead her students through an analysis and synthesis of a longer proof from the book. Moira did this by:

- Photocopying a proof from the text.
- Cutting the proof into pieces.
- Taping the pieces of the proof on a piece of paper, leaving space between the pieces.
- Giving each student two copies of the cut-and-paste page.

The students were then requested to:

- On one copy, fill in the details and missing steps (analysis).
- On the second copy, fill in the big picture and outline the structure of the proof (synthesis).

The students were then required to do a similar analysis and synthesis exercise to another proof from the book, this time creating their own photocopied cut-and-pasted proofs.

ENCOURAGING PARTICIPATION: ACTIVE STUDENTS OUTSIDE THE CLASSROOM

Although we would have all liked to assume the dynamic state of work and discussion in our classroom was naturally carried into the homes and study groups of our students, we also felt it would be beneficial to provide motivation for working outside of the classroom. The following outlines some of our techniques to inspire our students to be active learners at all times.

Example: Collaborative Homework

Moira required her students to work in groups on the homework. Students were divided into groups of three or four by the instructor and these groups were changed several times during the semester. Students worked together on the problems and then divided up the problems and shared the writing of solutions. For each homework assignment each group turned in one final version of their solutions. Additionally, each problem set contained one or two starred problems that each student had to write up independently. Moira found that the students actually did share the work. She also found the
students felt less pressure than if they had been required to do all of the problems by themselves and that they thought the writing burden was less onerous when assigned in this style. It is worth noting that Moira did think that some of her students would have benefited from more individual work.

Example: Student Papers

sarah-marie required a paper each term, with some criteria in common (the students were allowed to suggest topics).

The first term, she required a 2-3 page word-processed expository paper on a student-selected subject from the Special Topics section of [1]. The audience of the paper was specified to be the students’ class peers, and the topic had to be specified a month before the paper was due. Rough drafts were requested, but not required.

As it turned out, the students were quite inexperienced at writing mathematically and most submitted several rough drafts for comments. Some of them made corrections simply because they were requested, rather than because they understood that this would improve the paper. Even so, students who continued to the next term turned in more pleasingly-written homework and exams. They had improved their proof writing skills (some dramatically!) and general mathematical communication skills via this paper.

In the second term, sarah-marie modified the instructions. The rough draft was required, significantly before the final draft was due, to prevent students from doing shoddy work at the last minute. Furthermore, the students each had to give a 15-20 minute class presentations on their (now) 5-7 page papers. They were additionally instructed to create a handout to be given to the class at least one period prior to the presentation. Giving the presentations served three purposes: (1) to prevent the students from making corrections on their papers that they did not understand, (2) to give the students presentation practice, and (3) to give the students practice in distilling complex ideas for a general audience.

Finally, the students had to write sample problems and solutions for the final exam based on their respective topics.

The second term, the papers turned out better on the whole, with fewer rough drafts needed. At this point, the students were concentrating on learning the mathematics and presenting it well.

Moira also had her students write papers during the second semester. Her paper assignment differed from sarah-marie’s in that each student selected a different topic, chose reading material for the class, and eventually presented material on their topic to the class for two to three days. The students were generally creative in their class presentations. In addition to interactive lecturing, they created worksheets, group activities and computer exercises for the class. The papers were an outgrowth of the presentations and they were
uniformly of good quality, in part because the students had already spent a
good deal of time learning the material and preparing their class presenta-
tions. Most students submitted drafts of their papers. Their ability to write
about mathematics was noticeably better than it had been at the beginning
of the semester.

**Example: Term Projects**

Laurie wanted her graduate students to connect the abstractions of modern
algebra to activities and ideas they could bring into their high school class-
rooms. The graduate students were instructed to imagine they were creating
a special topics module (with unlimited time) for a promising group of high
school students.

The term project assignment was:

Write a series of lesson plans, lecture notes and worksheets:
- To guide the students in using manipulatives (triangle, square, ...) to
discover the properties of a group.
- To teach the students the definition of a group.
- To teach the students the very basics of modular arithmetic.
- To use the properties of a clock to model the behavior of \( \mathbb{Z}_{12} \) (this needs
to be a worksheet).
- To teach the students enough additional information to support the fol-
lowing project.

**Project for the (high school) students:**

- Gather at least 6 items with UPC barcodes on them, and then, using the
  information we have learned in class, check that the check digit is correct.

The graduate students turned in exceptional term projects. They fully
embraced the ideas behind the assignment and utilized their skills as mathematics
teachers to synthesize some of the ideas of their high level course into clear
concepts that they could share with their students.

**ACTIVE ASSESSMENT**

We all felt homework should be both designed and graded to foster quality proof writing
skills while also providing general encouragement to our students. We used widely varying
 techniques to achieve these goals.

**Example: Copious Homework Comments**

In contrast to on-the-board proof writing and in-class peer review, sarah-
marie's method for getting her students to write good proofs consisted entirely
of working with their homework. She wrote numerous comments on their
wording and grammar, and on whether statements were missing or, instead,
unnecessary. This resulted, in some cases, in returned homework containing
more of sarah-marie's writing than the student's. sarah-marie regularly pro-
vided her students with homework solutions so they would have examples to
work from and improve upon. When stopping by the Math Lab, sarah-marie
often discovered her students discussing which points were important to make in their proofs and how to phrase them so that they would be acceptable.

sarah-marie did not allow rewrites on the homework; she did this in order to encourage the students to try quite hard to turn in quality work. sarah-marie found that one of the results of her strict grading policies was that many of the students asked her to look over their proofs before they handed them in to be graded.

**Example: Rewriting Homework**

Laurie is a firm believer in the value of students rewriting selected homework problems until they are correct. She utilized this technique in both her graduate and undergraduate courses. Her grading procedure was simple; she would include brief but helpful comments by the homework problems (mostly on the proofs) that were poorly or illogically presented. The students were requested to rewrite the problem and resubmit the corrected paper (for a reduced score on the resubmitted problems). Most of the students were able to clearly understand, with a few comments, what their errors were and they were enthusiastic about the opportunity to rewrite the homework problems that they had originally missed. Laurie found the rewriting requirement to be a valuable aid in the instruction of her students; it helped them to improve their proof writing skills and it helped their retention of core material for the exams.

**Example: Take Home Exams**

Because writing proofs was an important part of all three of our courses, we all thought that is was important that our exams have some take-home component. Moira had originally planned to give in-class exams as well as take-home exams. In the end, we all gave only open-book take-home exams. Moira wanted to ask her students to prove things on the exams and didn’t want time to be an issue. Laurie’s exams were mostly book problems without hints. Her intent was that the questions would be fairly easy and accessible if the students had been paying attention and doing the homework. She found the sectional Supplementary Exercises in [1] to be a good source of exam questions. In contrast, sarah-marie gave more difficult exams, in the sense that she picked the hardest problems from the book. Her students spent forth to fifty hours on each exam, of which about ten were spent in sarah-marie’s office. It was an enormous time commitment, but both sarah-marie and her students thought that they learned a tremendous amount from the exams.

**OUR COLLABORATION**

From the beginning we found our collaboration to be supportive and inspirational. In Atlanta we were able to choose [1] as our text and actually confer with the author regarding our ideas for our courses. This helped us to feel like we were part of a strong collaborative effort instead of only feeling like less experienced individuals. Our discussions helped us to
refine our course designs and gain useful feedback regarding the practicality and usefulness
of our specific ideas. Throughout the year we asked each other a lot of questions, both on
content details and on how we planned to present different concepts. These discussions
were valuable when one member of our team had strength in a particular topic and always
generated useful ideas for all of us.

ACTIVE PROFESSORS
We all endeavored to organize our courses to elicit maximum participation, to emphasize
active learning, and to create an environment in which our students worked hard but found
the work rewarding and enjoyable. In order to accomplish these goals, we found it necessary
to be active professors to exemplify for our students what an active commitment to an
abstract algebra course is. As is shown by our various in class and out of class techniques,
we all provided a great deal of feedback both in our classrooms and in response to student
work. Furthermore, we tried to do this with an enthusiastic and positive attitude. As a
result, students spent a lot of time in our offices (during office hours and in some cases
out of office hours) analyzing the course material with us. Overall, we found the extra
encouragement and time we provided our students produced effective courses focused on
student success.
REFERENCES:


