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CHAPTER 1

INTRODUCTION

Mathematicians have long used the idea of reduction to characteristic p to solve problems in characteristic zero. More recently, Hochster made use of characteristic p techniques to solve several homological conjectures. Tight closure techniques exploit the Frobenius endomorphism in characteristic p and have proven to be a very powerful tool. Hochster's proof of the existence of big Cohen-Macaulay modules may be the first "tight closure" proof, although the idea was not mentioned explicitly. The idea was finally made explicit when Hochster and Huneke introduced the notion of tight closure for ideals of commutative Noetherian rings of characteristic p and for submodules of finitely generated modules over certain Noetherian rings. The definition is first made in characteristic p using the action of the Frobenius endomorphism. A notion for finitely generated algebras over a field of characteristic zero is then obtained by reduction to characteristic p. There is not yet a satisfactory notion in mixed characteristic.

Tight closure techniques have been used to give new proofs, often in greatly strengthened form, of the following seemingly unrelated results: rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay, the integral closure of the nth power of an n generator ideal of a regular ring is contained in the ideal (the Briançon-Skoda theorem), the monomial conjecture and the syzygy theorem [HH1]. Hochster and Huneke [HH3] also used techniques inspired by tight closure to prove that if R is an excellent domain of characteristic p, then R^+, the integral closure of R in an algebraic closure of its fraction field, is a balanced big Cohen-Macaulay algebra.

Some of the important features of tight closure include the fact that every ideal in a regular ring is tightly closed and the tight closure of an ideal is contained in the integral closure of the ideal. In addition, if R is a local ring and $x_1, \dots, x_n$ are part of a system of parameters of R, then the ideal $(x_1, \dots, x_{n-1})\colon x_n$ is contained in the tight closure of $(x_1, \dots, x_{n-1})$ under mild conditions on R. In a sense, tight closure captures the failure of a ring to be Cohen-Macaulay.

Despite the apparent power of tight closure techniques, the tight closure operation itself is quite difficult to handle in practice. For example, it is generally difficult to find the tight closure of an arbitrary ideal. Also, it is not known whether tight closure behaves well under localization. We would like to know whether it is true that $I^*W^{-1}R = (IW^{-1}R)^*$ where I is an ideal of a ring R, W is an arbitrary multiplicative system and I^* denotes the tight closure of I. It is not even known that if all of the ideals of R are tightly closed, then all of the ideals of R_P are tightly closed. In light of these difficulties, it is of great interest to describe tight closure in different ways. One approach is to answer positively the conjecture that the tight closure of an ideal I, I^*, is the same as the expansion and contraction of I from R to R^+, $IR^+ \cap R$.

Hochster and Huneke have shown that if R is an excellent domain of characteristic

p, then R^+ is a big Cohen-Macaulay algebra for R [HH3]. This result, combined with the results of [HH6], imply that if I is an ideal of R, then $IR^+ \cap R \subseteq I^*$. It is an open question whether or not $IR^+ \cap R$ is in fact equal to I^*. If so, it would follow that tight closure does commute with localization. In Chapters 4 and 6 we will show that $I^* = IR^+ \cap R$ for certain classes of ideals in cubical cones.

One of the things that makes tight closure particularly difficult to compute in practice is the fact that one must check certain conditions for possibly infinitely many values of $q = p^e$. It would be helpful if there were a bound on the power of p necessary to check whether an element is in the tight closure of an ideal. Not only would the existence of a bound make it feasible to calculate tight closure, but the existence of such a bound would solve the localization problem. In Chapter 3 we discuss a mapping property that implies the existence of a bound and show that certain rings have this property.

Although it is usually very difficult to determine the tight closure of an ideal, sometimes calculations can provide evidence that an element is or is not in the tight closure of a given ideal. Much of the work behind this thesis was inspired by calculations made using the computer package *Macaulay*.

An Outline of the Thesis

Chapter 2 gives a brief review of the definitions and notational conventions used in this work as well as an overview of some of the basic tight closure results.

In Chapter 3 we describe a mapping property for rings of characteristic p which implies a bound on the value of q necessary to test whether an element is in the tight closure of an ideal. We also show that the mapping property implies that tight closure commutes with localization. The main result in this chapter is a proof that one-dimensional F-finite rings have this mapping property.

In Chapter 4 we consider tight closure, plus closure and Frobenius closure in rings of the form $K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \neq 3$. These rings can be thought of as the homogeneous coordinate rings of elliptic curves. We describe a $\mathbb{Z}_3$-grading on these rings and show how we can use the grading to reduce questions about ideals in the coordinate rings to questions about ideals in the regular ring $K[[x, y]]$. Using these techniques we show that Frobenius closure is the same as tight closure in certain classes of ideals when $p \equiv 2 \mod 3$. Using the fact that $I^F \subseteq IR^+ \cap R \subseteq I^*$, we conclude that $IR^+ \cap R = I^*$ for these ideals.

In Chapter 5 we discuss injective modules over the non-Noetherian ring R^∞, the union of all p^eth roots of elements of R, where again $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$. Using these injectives we prove that one can reduce the question of whether $I^F = I^*$ for $\mathbb{Z}_3$-graded ideals to the case of $\mathbb{Z}_3$-graded irreducible modules.

In Chapter 6, motivated by the results of Chapter 5, we study the case of irreducible m-primary $\mathbb{Z}_3$-graded ideals in $K[[x, y, z]]/(x^3+y^3+z^3)$. We use the grading developed in Chapter 4 to classify the irreducible $\mathbb{Z}_3$-graded ideals. We then show that $I^F = I^*$ for most irreducible $\mathbb{Z}_3$-graded ideals in $K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Hence $I^* = IR^+ \cap R$ for these ideals.

In Chapter 7 we discuss yet another closure operation, regular closure. The regular closure contains the tight closure when both are defined, but in general, the regular closure is strictly larger. Regular closure is interesting, in part, because it is defined *a priori* in all characteristics, including mixed characteristic. We show that one can test regular closure in R by only considering local maps to regular local rings. In certain cases, it is necessary only to consider maps to certain affine algebras. We also prove the equivalence of two variants of regular closure for a class of rings that includes $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$.

CHAPTER 2

TIGHT CLOSURE

In this chapter we provide a brief introduction to the theory of tight closure. We are primarily interested in the case where tight closure is an operation performed on ideals in a commutative Noetherian ring of characteristic p. Tight closure is also defined for submodules of modules over a Noetherian ring. In addition there are several notions of tight closure in equal characteristic zero which involve reduction mod p. It is still unclear how to define tight closure in an effective way in mixed characteristic.

The definition of the tight closure operation involves iterating the Frobenius endomorphism of a ring. Throughout this chapter, R denotes a commutative Noetherian ring of characteristic p and q always denotes some prime power, p^e, for some non-negative integer e.

2.1 Notations and Conventions

We denote by F or F_R the Frobenius endomorphism of a ring R of characteristic p, and we denote by F^e the eth iteration of F, so that $\mathrm{F}^e(r) = r^q$. If R is a reduced ring of characteristc p, we write $R^{1/q}$ for the ring obtained by adjoining qth roots of all elements of R. The inclusion map $R \hookrightarrow R^{1/q}$ is isomorphic with $\mathrm{F}^e : R \to R$. We write R^∞ for $\cup_q R^{1/q}$. Note that R^∞ is almost never Noetherian.

If $I \subseteq R$ and $q = p^e$, then $I^{[q]}$ denotes the ideal $(i^q : i \in I)R$, which is also the expansion $\mathrm{F}(I)R$ of I under the Frobenius map $\mathrm{F} : R \to R$. Note that if T denotes a set of generators for R, then $\{\, t^q : t \in T \,\}$ generates $I^{[q]}$.

In any commutative ring R, R° denotes the complement of the union of the set of minimal primes. Thus, if R is a domain, $R^\circ = R \backslash \{0\}$.

(2.1.1) Definition. An injection of R-modules $M \to N$ is called *pure* if $M \otimes_R W \to N \otimes_R W$ is an injection for all R-modules W. An extension of rings $R \subseteq S$ is *pure* if it is pure as an extension of R-modules.

(2.1.2) Definition. A local ring $(R,\ m,\ k)$ of dimension d is *Gorenstein* if it is Cohen-Macaulay and for some (equivalently every) system of parameters, if I denotes the ideal generated by the parameters, then $\dim_k(\mathrm{Ann}_{R/I} m) = 1$. The module $(\mathrm{Ann}_{R/I} m)$ is called the *socle* of R/I, and its dimension as a k-vector space is called the *type* of R.

Hence Gorenstein rings are precisely those Cohen-Macaulay rings of type 1. In the global case, R is called *Gorenstein* if and only if for each maximal ideal m, the localized ring R_m is Gorenstein. Gorenstein local rings can also be characterized as those Noetherian rings having finite injective dimension when viewed as modules over themselves [Mat].

2.2 Tight Closure

(2.2.1) Definition. Let R be a ring of characteristic p and I an ideal of R. We say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$. If $I = I^*$, we say that I is *tightly closed.*

(2.2.2) Discussion. Let R be a ring of characteristic p and $N \subseteq M$ finitely generated R-modules. We can map a finitely generated free module G onto M, say $f\colon R^h \twoheadrightarrow M$. Let $H = f^{-1}(N)$ and let $v \in R^h$ map to $u \in M$. Then $u \in N^*_M$, the *tight closure of N in M*, if and only if $v \in H^*_{R^h}$. In the free case, the definition of tight closure is exactly the same as for ideals. We set $(r_i, \ldots, r_h)^q = (r_1^q, \ldots, r_h^q)$ and let $H^{[q]}$ be the R-span in M of the elements h^q for $h \in N$. A more functorial approach to tight closure in modules is given in [HH1], for example.

We now outline some basic properties of tight closure.

(2.2.3) Proposition. *Let R be a Noetherian ring of characteristic p, and let I, J be ideals of R.*

- (a) *I^* is an ideal of R containing I.*
- (b) *$(I^*)^* = I^*$.*
- (c) *If $I \subseteq J$, then $I^* \subseteq J^*$.*
- (d) *If I has positive height or if R is reduced, then $x \in I^*$ if and only if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e$.*
- (e) *For any $I \subseteq R$, I^* is the inverse image in R of $(IR_{\mathrm{red}})^*$.*
- (f) *If I is tightly closed, then $I : J$ is tightly closed for any ideal J.*
- (g) *$I^* \subseteq \bar{I}$.*
- (h) *An element $x \in R$ is in I^* if and only if the image of x in R/P is in the tight closure of $(I + P)/P$ for every minimal prime P of R.*

Proof. See [HH1]. (a)-(f) can be found in Proposition 4.1. (g) is Theorem 5.2, and (h) is Proposition 6.25(a).

(2.2.4) Remark. Virtually all of the theory of tight closure reduces to the case where R is a reduced ring. If N denotes the nilradical of R and $I \subset R$, then $u \in I^*$ if and only if $\bar{u} \in (I/N)^*$ where $\bar{u}$ is the image of u in I/N. Also note that Proposition 2.2.3 (h) means that we can reduce to the domain case.

2.3 Test Elements

In many applications one would like to be able to choose the element c in the definition of tight closure independent of x or I. It is very useful when a single choice of c, a test element, can be used for all tight closure tests in a given ring.

The theory of test elements has turned out to be very important in tight closure theory. For example, the existence of test elements is used to prove that tight closure persists under homomorphisms, i.e., given $I \subseteq R$ and $z \in I^*$, then $\phi(z) \in (IS)^*$, where $\phi : R \to S$ is a homomorphism. Also, from a computational point of view, test elements make it easier to show that an element is not in the tight closure of a given ideal. If c is a test element and $cu^q \notin I^{[q]}$ for *one* q, then $u \notin I^*$.

(2.3.1) Definition. An element $c \in R^\circ$ is a *test element* for R if for all ideals $I \subseteq R$ and all $u \in R$, if $u \in I^*$, then $cu^q \in I^{[q]}$ for all q (equivalently $cu \in I$). We say that an element of R is a *completely stable test element* if its image in every local ring of R is a test element and its image in the completion of every local ring of R is a test element.

Fortunately, completely stable test elements exist for the rings we will consider.

(2.3.2) Theorem. *Let R be a reduced excellent local ring. If $c \in R^\circ$ has the property that R_c is regular, then c has a power which is a completely stable test* *element for R. In particular, completely stable test elements always exist for reduced excellent local rings.*

Proof. [HH5, Theorem 6.2]

When R has a test element, the ideal $\tau(R)$ generated by the test elements is an interesting invariant of R. All the elements of $\tau(R)$ not in a minimal prime are test elements. One can define $\tau(R)$ in a way that makes sense whether R has a test element or not [HH1, Definition 8.22]. However, since the rings we are interested in all have test elements, we will make the following intuitive definition.

(2.3.3) Definition. The ideal of all $c \in R$ such that for any ideal $I \subseteq R$, we have $cu^q \in I^{[q]}$ for all q whenever $u \in I^*$ is called the *test ideal* for R, denoted $\tau(R)$. The ideal of all $c \in R$ such that for all parameter ideals (see (2.4.2)) $I \subseteq R$, we have $cu^q \in I^{[q]}$ for all q whenever $u \in I^*$ is called the *parameter test ideal* for R.

2.4 F-regular and F-rational Rings

Rings in which all ideals or all parameter ideals are tightly closed have many interesting properties.

(2.4.1) Definition. A commutative Noetherian ring of prime characteristic is said to be *weakly F-regular* if all ideals are tightly closed. A Noetherian ring R such that $W^{-1}R$ is weakly F-regular for every multiplicative system W is called *F-regular*.

It is awkward to have the terminology “weakly” F-regular. If tight closure commutes with localization, then these two definitions are equivalent.

We know that weakly F-regular implies F-regular for Gorenstein rings [HH5], for algebras finitely generated over an uncountable field of characteristic p [HH5], and, with mild assumptions, for rings of dimension at most 3 [W].

(2.4.2) Definition. A commutative Noetherian ring of prime characteristic is said to be *F-rational* if all parameter ideals are tightly closed. (An ideal generated by i elements is said to be a *parameter ideal* if it has height at least i.)

We now summarize a few useful facts about F-regular and F-rational rings.

(2.4.3) Proposition.

(a) *A regular ring of characteristic p is F-regular.*

(b) *R is weakly F-regular if and only if R_m is weakly F-regular for all maximal ideals m of R.*

(c) *If R_m is F-rational for all maximal ideals of R, then R is F-rational.*

(d) *An excellent local ring (R, m) is F-rational if and only if some ideal generated by some (full) system of parameters is tightly closed.*

(e) *If a locally excellent ring R is F-rational, then so is $W^{-1}R$, where W is any multiplicative system in R.*

(f) *A Gorenstein ring is weakly F-regular if and only if it is F-rational.*

Proof. (a) Theorem 4.4 of [HH1].

(b) Corollary 4.15 of [HH1].

(c), (d) and (e) Proposition 1.4.3(ii), (v) and (vii) of [S1]. See also Theorem 4.2(c), (d) and (f) of [HH5] for the case when R is a homomorphic image of a Cohen-Macaulay ring.

(f) Theorem 4.2(g) of [HH5]. $\square$

We will have reason to study yet another class of rings in Chapter 3. This property, strong F-regularity, is only defined for reduced rings R such that $R^{1/p}$ is module-finite over R. However, finitely generated algebras over a perfect field K and complete local rings R with a perfect residue class field K both satisfy this condition, so it is not particularly limiting. Again, it may be true that strong F-regularity is the same as F-regularity for rings in which both are defined. This is not known in general, but there is evidence of this in low dimension in [W].

(2.4.4) Definition. A Noetherian domain R of characteristic p is *strongly F-regular* if $R^{1/p}$ is a finite R-module and if for every nonzero $d \in R$, there exists a q such that the map $R \to R^{1/q}$ sending 1 to $d^{1/q}$ splits as a map of R-modules.

We now summarize a few useful facts about strongly F-regular rings.

(2.4.5) Proposition. *Let R be a reduced Noetherian ring of characteristic p such that $R^{1/p}$ is module-finite over R.*

- (a) *R is strongly F-regular if and only if R_P is strongly F-regular for every prime (respectively, for every maximal) ideal P of R. Hence, if R is strongly F-regular, so is $W^{-1}R$ for every multiplicative system W.*
- (b) *If R is regular, then R is strongly F-regular.*
- (c) *If R is strongly F-regular, then R is F-regular.*
- (d) *If R is weakly F-regular and Gorenstein, then R is strongly F-regular.*

Proof. See [HH5, Theorem 5.5]

2.5 Localization

We do not know, in general, how tight closure behaves under localization. In particular, we would like to know the answer to the following question: given an ideal, I, of a ring, R, and an arbitrary multiplicative system W, is it true that $I^*W^{-1}R = (IW^{-1}R)^*$? The localization problem has been solved in a number of special cases. For example, localization at a maximal ideal behaves as desired.

(2.5.1) Proposition. *Let R be a Noetherian ring of characteristic p, and let I be an ideal primary to a maximal ideal m. Then $(IR_m)^* = I^*(R_m)$. Hence, if I is tightly closed, so is IR_m. Moreover, I^* is the contraction of $(IR_m)^*$.*

Proof. See Proposition 4.14 of [HH1]

Also, if I is generated by parameters in a locally excellent locally equidimensional ring R, then $I^*W^{-1}R = (IW^{-1}R)^*$ for any multiplicative system W of R [AHH]. In addition, we also know that tight closure commutes with localization for $N \subseteq M$ modules such that M/N has a finite phantom projective resolution [AHH].

CHAPTER 3

MAPPING PROPERTIES AND LOCALIZATION

In this chapter, we examine certain mapping properties of rings of characteristic p. On the face of it, in order to check whether an element is contained in the tight closure of an ideal, we must check whether a certain condition holds for possibly infinitely many powers of p. We would like to know whether we can check this condition for only finitely many values of q where $q = p^e$. The existence of such a bound is one of many conditions that implies that tight closure commutes with localization. Here, we shall study a mapping property that implies the existence of such a bound.

The main result in this chapter is Theorem 3.3.4 which establishes that the mapping property holds for one-dimensional F-finite domains. It seems doubtful that these maps exist in higher dimensions in general as that would imply the existence of certain differential operators which do not seem to exist. But the question remains quite open. Moreover, there is some evidence, particularly in $K[[x,y,z]]/(x^3+y^3+z^3)$ when the characteristic of K is two, that a bound exists on the power of p needed to test for tight closure. Thus, it may be that there is a weaker property which holds in general and implies the existence of a bound.

The mapping property we will be discussing is the following: given a test element

c, we call R *strongly bounded relative to* c if there exists an R-linear map $\theta : R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$ for some q, and we call q the *bounding exponent.* Note that the map θ and the exponent q both depend on c.

First we will show that if a ring is strongly bounded relative to a test element, then tight closure commutes with localization in that ring. We will then show that in certain rings there are test elements such that the ring is strongly bounded relative to those test elements.

3.1 Localization

Before proving that the mapping property implies that tight closure commutes with localization, we need the following lemma.

(3.1.1) Lemma. *Let* R *be a reduced ring of characteristic* $p > 0$ *and* $c \in R^{\circ}$. *Suppose that* R *is strongly bounded relative to* c *with bounding exponent* q *for some* q. *Then* R *is strongly bounded relative to* c *with bounding exponent* q' *for all* $q' \geq q$.

Proof. Suppose there exists a map $\theta\colon R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$. We can define a map $\tilde{\theta}\colon R^{1/pq} \to R^{1/p^2q}$ with the desired properties. Let $\tilde{\theta}(r^{1/pq}) = [\theta(r^{1/q})]^{1/p}$. First we check that $\tilde{\theta}(c^{1/pq}) = c^{1/p^2q}$. By construction

$$\tilde{\theta}(c^{1/pq}) = [\theta(c^{1/q})]^{1/p} = (c^{1/pq})^{1/p} = c^{1/p^2q}.$$

Next we check that $\tilde{\theta}$ is R-linear. Let $x^{1/p} \in R^{1/p}$. Then $\tilde{\theta}(x^{1/p}) = [\theta(x)]^{1/p}$. But $x \in R$ and θ is R-linear, so $\theta(x) = x$ and hence $\tilde{\theta}(x^{1/p}) = x^{1/p}$. Thus $\tilde{\theta}$ is $R^{1/p}$-linear and hence R-linear. $\square$

We can now prove that the existence of these maps implies that tight closure commutes with localization.

(3.1.2) Proposition. *(a) Let R be a reduced Noetherian ring of characteristic p such that R is strongly bounded relative to a test element c with bounding exponent q. Let $I \subseteq R$ be an ideal and $u \in R$. Then $u \in I^*$ if and only if $cu^q \in I^{[q]}$.*

(b) Let R be a reduced Noetherian ring of characteristic p and let c be a test element. Suppose there exists q and an R-linear map $\theta\colon R^{1/q} \to R^{1/pq}$ such that $\theta(c^{1/q}) = c^{1/pq}$. Let I be an ideal of R and W a multiplicative system. Then $(W^{-1}I)^* = W^{-1}I^*$.

Proof. In the proof of (a), $\implies$ is clear. For the other direction, if $cu^q \in I^{[q]}$, taking qth roots gives $c^{1/q}u \in IR^{1/q}$. Apply the map $\theta\colon R^{1/q} \to R^{1/pq}$ to get $c^{1/pq}u \in IR^{1/pq}$. This implies that $cu^{pq} \in I^{[pq]}$. Repeating the argument using Lemma 3.1.1 shows that $cu^{q'} \in I^{[q']}$, all $q' \geq q$. Hence $u \in I^*$.

To prove (b), first note that it suffices to see that $(W^{-1}I)^* \subseteq W^{-1}I^*$. Pick $u \in (W^{-1}I)^*$. So $cu^q \in (W^{-1}I)^{[q]} = W^{-1}I^{[q]}$. We can pick $f \in W$ so that $fcu^q \in I^{[q]}$. This implies that $f^qcu^q \in I^{[q]}$. So c$(fu)^q \in I^{[q]}$. By part (a), $fu \in I^*$, and so $u \in W^{-1}I^*$. $\square$

3.2 Some Reductions

Before proving the main result in this chapter, we will discuss several reductions in the problem. The following two lemmas establish that the issue of giving a map $R^{1/q} \to R^{1/pq}$ is local and unaffected by completion.

(3.2.1) Lemma. *(a) Let R be a Noetherian ring. Let M and N be R-modules with M finitely presented, $v \in M$ and $w \in N$. Suppose for all $P \in \operatorname{Spec} R$ there exists a map $\phi_P \colon M_P \to N_P$ (R-linear or R_P-linear) with $\phi_P(v/1) = w/1$. Then there exists an R-linear map $\phi \colon M \to N$ such that $\phi(v) = w$.*

(b) Let R be a reduced F-finite Noetherian ring. Let c be a stable test element for R. Suppose R_P is strongly bounded relative to c with bounding exponent q for all $P \in \operatorname{Spec}(R)$. Then R is strongly bounded relative to c with bounding exponent q.

Proof. (a) Let $J = \{\, \psi(v) : \psi \in \operatorname{Hom}_R(M, N) \,\} \subseteq N$. If $w \notin J$, we can choose P such that $P \supseteq J \colon w$. Then $w/1 \notin J_P$. Since M is finitely presented, Hom commutes with localization. So $J_P = \{\, \phi(v/1) \colon \phi \in \operatorname{Hom}_{R_P}(M_P, N_P) \,\}$. Thus $w/1 \notin \{\, \phi(v/1) \colon \phi \in \operatorname{Hom}_{R_P}(M_P, N_P) \,\}$.

(b) Since R is F-finite, $R^{1/q}$ is module-finite over R. So $R^{1/q}$ is Noetherian and thus finitely presented as an R-module. The result now follows from (a). □

(3.2.2) Lemma. *Let R be a local ring, M and N finitely generated R-modules, $v \in M$ and $w \in N$. Suppose there exists a map $\hat{\phi} \colon \hat{M} \to \hat{N}$ such that $\hat{\phi}(\tilde{v}) = \tilde{w}$ where $\tilde{v}$ and $\tilde{w}$ are the images of v and w in $\hat{R}$. Then there exists a map $\phi \colon M \to N$ such that $\phi(v) = w$.*

In particular, if R is a reduced local F-finite ring and c is a stable test element for R, then R is strongly bounded relative to c if and only if $\hat{R}$ is strongly bounded relative to $\tilde{c}$ where $\tilde{c}$ is the image of c in $\hat{R}$.

Proof. Let $J = \{\, \phi(v) \colon \phi \in \operatorname{Hom}(M, N) \,\}$. If $w \notin J$, then there exists t such that

$w \notin J + m^t N$. Since R is Noetherian and M and N are finitely generated, we have

$$\mathrm{Hom}_{\hat{R}}(\hat{M}, \hat{N}) \cong \hat{R} \otimes_R \mathrm{Hom}_R(M, N) \cong \widehat{\mathrm{Hom}_R(M, N)}.$$

If we think of $\hat{\phi} \in \widehat{\mathrm{Hom}_R(M, N)}$, then we can find $\theta \in \mathrm{Hom}_R(M, N)$ such that $\hat{\theta}$, the image of θ in $\widehat{\mathrm{Hom}_R(M, N)}$, is within m^t of $\hat{\phi}$. Then $\hat{\phi}(\tilde{v}) - \hat{\theta}(\tilde{v}) \in m^t \hat{N}$, which implies that $\tilde{w} \in \hat{\theta}(\tilde{v}) + m^t \hat{N}$. So $\tilde{w} \in \tilde{J} + m^t \tilde{N}$, and hence $w \in J + m^t N$, which is a contradiction.

To see the last statement in the lemma, we first show that $\widehat{F^e(R)} \cong F^e(\hat{R})$. We know that $F^e(R) \cong R$, so it follows that $\widehat{F^e(R)} \cong \hat{R}$. Also, $F^e(\hat{R}) \cong \hat{R}$. Hence $\widehat{F^e(R)} \cong F^e(\hat{R})$ since both are isomorphic to $\hat{R}$.

Since R is reduced, we know that $F^e(R) \cong R^{1/q}$ and hence $\widehat{F^e(R)} \cong \widehat{R^{1/q}}$. As R is F-finite, R is excellent and hence $\hat{R}$ is reduced. Thus $F^e(\hat{R}) \cong \hat{R}^{1/q}$. Combining the previous results we see that $\widehat{R^{1/q}} \cong \hat{R}^{1/q}$. Thus $\mathrm{Hom}_{\hat{R}}(\hat{R}^{1/q}, \hat{R}^{1/pq}) \cong \mathrm{Hom}_{\hat{R}}(\widehat{R^{1/q}}, \widehat{R^{1/pq}})$. As R is F-finite, $R^{1/q}$ is module-finite over R, and the result now follows from the first part of the lemma. $\square$

3.3 Strongly Bounded Rings

Our first example of strongly bounded rings is strongly F-regular rings.

(3.3.1) Proposition. *Let R be a strongly F-regular ring and let c be a test element. Then there exists an R-linear map $R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$ for some q. In other words, a strongly F-regular ring is strongly bounded relative to every test element c.*

Proof. Since R is strongly F-regular, we know that the R-linear map $R \to R^{1/q}$ that sends 1 to $c^{1/q}$ splits as a map of R-modules. Let θ be the splitting map. Then if we compose θ with multiplication by $c^{1/pq}$, we have

$$\begin{array}{ccccc} R^{1/q} & \xrightarrow{\theta} & R & \xrightarrow{c^{1/pq}} & R^{1/pq} \\ c^{1/q} & \longrightarrow & 1 & \longrightarrow & c^{1/pq} \end{array}$$

and the composition is R-linear. □

It is also quite easy to see that the desired maps exist for a one-dimensional domain R when the integral closure of R is contained in $R^{1/q}$ for some q.

(3.3.2) Proposition. *Let R be a one-dimensional F-finite domain and let S be the integral closure of R in its fraction field. Let c be a completely stable test element for R. Suppose, for large q, $S \subseteq R^{1/q}$. Then there exists a q and a map $\theta\colon R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$.*

Proof. We may pass to the local case by Lemma 3.2.1. By assumption we have that $S \subseteq R^{1/q}$ for $q \gg 0$. Since normal implies regular in dimension one and a one-dimensional regular domain is a principal ideal domain, S is a PID. As R is a domain, $R^{1/q}$ is torsion-free as an S-module. Finitely generated torsion-free modules over a PID are free, so $R^{1/q}$ is free as an S-module. We can give an S-linear map $R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ wherever we like if $c^{1/q}$ is part of an S-free basis for $R^{1/q}$. In the local case a free basis is just a minimal basis, so $c^{1/q}$ is part of an S-free basis if and only if $c^{1/q} \notin m_S R^{1/q}$, and this is true if and only if $c \notin {m_S}^{[q]}R$. Since R is one-dimensional, m_R contains some power of $m_S R$. To see this, note that since S is an integral extension of R in its fraction field, we can pick $b \in R\backslash\{0\}$ such that $bS \subseteq R$. In fact, $bS \subseteq m_R$. The ideal bS is a nonzero ideal of R and since R is one-dimensional, m_R is the radical of bS. We can pick q_0 such that ${m_S}^{[q_0]} \subseteq bS \subseteq m_R$. Pick q_1 so large that $c \notin {m_R}^{[q_1]}$ and let $q = q_0 q_1$. Then $c \notin {m_S}^{[q]}R = {m_S}^{[q_0 q_1]}R$ since ${m_S}^{[q_0 q_1]}R \subseteq ({m_S}^{[q_0]})^{[q_1]} \subseteq {m_R}^{[q_1]}$ and q_1 was chosen precisely so that $c \notin {m_R}^{[q_1]}$. $\square$

Next we give an example of a simple case when the integral closure of R is not contained in $R^{1/q}$, and we show that R is strongly bounded.

(3.3.3) Example. *Let* $R = K + K(\lambda)x + K(\lambda)x^2 + K(\lambda)x^3 + \cdots$, *where* λ *is separable over* K*. Then there exists an* R*-linear map* $R^{1/q} \to R^{1/pq}$ *taking* $x^{1/q}$ *to* $x^{1/pq}$*.*

Proof. Let S be the integral closure of R, so $S = K(\lambda) + K(\lambda)x + K(\lambda)x^2 + \cdots$. $R \subseteq S$ extends to injective R-linear maps $i^{1/q}\colon R^{1/q} \to S^{1/q}$ and $i^{1/pq}\colon R^{1/pq} \to S^{1/pq}$. Since S is integrally closed, we can apply (3.3.2) to get an S-linear map $\theta\colon S^{1/q} \to S^{1/pq}$ taking $x^{1/q}$ to $x^{1/pq}$. We will construct θ explicitly later. So far we have the following maps:

$$\begin{tikzcd}
R^{1/q} \arrow[r, "i^{1/q}"] & S^{1/q} \arrow[r, "\theta"] & S^{1/pq} \\
 & & R^{1/pq} \arrow[u, "i^{1/pq}"']
\end{tikzcd}$$

Let $\phi = \theta \circ i^{1/q}$. It is enough to show that $\phi(R^{1/q}) \subseteq i^{1/pq}(R^{1/pq})$, for then the desired map is ϕ followed by restriction to $R^{1/pq}$.

Let $1 = \mu_0, \mu_1, \ldots, \mu_h$ be a basis for $K^{1/q}$ over K. Since $K(\lambda)/K$ is separable and $K^{1/q}/K$ is purely inseparable, the two extensions are linearly disjoint. So $1, \mu_1, \ldots, \mu_h$ is a basis for $K(\lambda)^{1/q}$ over $K(\lambda)$. Thus $\{\mu_i x^{j/q}\}_{0 \le i \le h}^{0 \le j < q}$ is a free basis for $S^{1/q}$ over S. Define an S-linear map $S^{1/q} \to S^{1/pq}$ by:

$$\theta(\mu_i x^{j/q}) = \begin{cases} x^{1/pq}, & i = 0 \text{ and } j = 1 \\ 0, & \text{otherwise} \end{cases}$$

$R^{1/q}$ is generated over R by $\{\mu_i \lambda^{t/q} x^{s/q} \colon {\scriptstyle 0 \le i \le h, 0 \le t < q, 0 < s < q}\}$. We can find $\phi(R^{1/q})$ by looking at what θ does to the generators of $R^{1/q}$.

Since $\mu_i \lambda^{t/q} \in K(\lambda)^{1/q}$ we can express it in terms of the basis for $K(\lambda)^{1/q}$.

$$\mu_i \lambda^{t/q} x^{s/q} = \sum_{j=0}^{h} \nu_{itj} \mu_j x^{s/q} \qquad \text{where } \nu_{itj} \in K(\lambda)$$

Since $\theta(\mu_j x^{s/q}) = 0$ for $j \neq 0$, and $\nu_{itj} \in K(\lambda) \subseteq S$,

$$\theta(\mu_i \lambda^{t/q} x^{s/q}) = \begin{cases} \nu_{it0} x^{s/pq}, & s = 1 \\ 0, & \text{otherwise} \end{cases}$$

Recall that

$$S^{1/pq} = K(\lambda)^{1/pq} + K(\lambda)^{1/pq} x^{1/pq} + \cdots + K(\lambda)^{1/pq} x^{pq-1/pq} + K(\lambda)^{1/pq} x + \cdots .$$

So $\nu_{it0} x^{s/pq} \in S^{1/pq}$ since $\nu_{it0} \in K(\lambda) \subseteq K(\lambda)^{1/pq}$. Then $\theta(R^{1/q}) \subseteq i^{1/pq}(R^{1/pq})$ as required. $\square$

We now prove the main result of this chapter.

(3.3.4) Theorem. *Let R be a one-dimensional F-finite domain and let S be the integral closure of R. For all $c \in R \backslash \{0\}$ such that $cS \subseteq R$, there exists $q = p^e$ and a map $\theta \colon R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$.*

Proof. We may reduce to the complete local case by Lemmas 3.2.1 and 3.2.2. So we are done by the following Proposition. $\square$

(3.3.5) Proposition. *Let R be a one-dimensional complete local domain. Let S be the integral closure of R. For any $c \in R\backslash\{0\}$ such that $cS \subseteq R$, there exists a q and an R-linear map $R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$.*

Proof. $R \subseteq S$ extends to an injective R-linear map $i^{1/q}\colon R^{1/q} \to S^{1/q}$. Let i be the inclusion map. Then $i^{1/q}(r^{1/q}) = (i(r))^{1/q}$.

For large enough q, $c^{1/q}$ is part of a free S-basis for $S^{1/q}$. Since S is regular, the Frobenius endomorphism is flat and so $S^{1/pq}$ is $S^{1/q}$-free. Hence there is an S-linear map $\theta\colon S^{1/q} \to S^{1/pq}$ taking $c^{1/q}$ to 1.

Multiplication by $c^{1/pq}$ gives a map $S^{1/pq} \to R^{1/pq}$. Composing $i^{1/q}$, θ, and multiplication by $c^{1/pq}$ gives the required map.

$$\begin{tikzcd}
R^{1/q} \arrow[r, "i^{1/q}"] & S^{1/q} \arrow[r, "\theta"] & S^{1/pq} \arrow[r, "c^{1/pq}"] & R^{1/pq} \\
c^{1/q} \arrow[r] & c^{1/q} \arrow[r] & 1 \arrow[r] & c^{1/pq}
\end{tikzcd}$$

Note that $cS \subseteq R$ implies $c^{1/pq}S^{1/pq} \subseteq R^{1/pq}$, so $c^{1/pq}\theta i^{1/q}$ is a map $R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$. The map $i^{1/q}$ is R-linear, θ is S-linear, and multiplication by $c^{1/pq}$ is $S^{1/pq}$-linear, so the composition is at least R-linear. $\square$

(3.3.6) Remark. Let R be a domain and S the integral closure of R in its fraction field. Let $J = \{\, j \in R : jS \subseteq R \,\}$, the conductor of S into R. J is an ideal of R, and if R is one-dimensional, any non-zero, non-unital element of R has a power in J. In particular, any test element for R has a power in J. A power of a test element is still a test element, so we can always pick c in Theorem 3.3.4 to be a test element. In other words, if R is a one-dimensional F-finite domain and c is a test element, then there exists an integer N such that R is strongly bounded relative to c^N.

Next we show that Proposition 3.3.5 holds for reduced rings.

(3.3.7) Proposition. *Let R be a one-dimensional complete local reduced ring, and let S be the integral closure of R in its total quotient ring. For any $c \in R\backslash\{0\}$ such that $cS \subseteq R$, there exists q and an R-linear map $R^{1/q} \to R^{1/pq}$ taking $c^{1/q}$ to $c^{1/pq}$.*

Proof. Let $p_1, \dots, p_k$ be the minimal primes of R. Since R is reduced, the integral closure of R in its total quotient ring is just $\prod_{i=1}^{k}(R/p_i)'$, where $(R/p_i)'$ denotes the normalization of R/p_i. Note that $c(R/p_i)' \subseteq R/p_i$ for all i. Since each $(R/p_i)'$ is a one-dimensional normal domain, there exists q_i such that c^{1/q_i} is part of a free $(R/p_i)^{1/q_i}$-basis. Let $q = \max\{q_i\}$. So we have maps $\theta_i\colon (R/p_i)^{1/q} \to (R/p_i)^{1/pq}$ taking $c^{1/q}$ to 1. We construct $\theta\colon \prod(R/p_i)^{1/q} \to \prod(R/p_i)^{1/pq}$ componentwise:

$$\theta((r_i, \dots, r_k)) = (\theta_1(r_1), \dots, \theta_k(r_k)).$$

Multiplication by $c^{1/pq}$ gives a map $S^{1/pq} = (\prod(R/p_i))^{1/pq} \to R^{1/pq}$, thus we have the following maps:

$$\begin{tikzcd}
(\prod R/p_i)^{1/q} \arrow[r, "\cong"] & \prod(R/p_i)^{1/q} \arrow[r, "\theta"] & \prod(R/p_i)^{1/pq} \arrow[r, "\cong"] & (\prod R/p_i)^{1/pq} \arrow[d, "c^{1/pq}"] \\
R^{1/q} \arrow[u] & & & R^{1/pq}
\end{tikzcd}$$

The composition of these maps takes $c^{1/q}$ to $c^{1/pq}$ and is R-linear. $\square$

We can also show that polynomial rings over strongly bounded rings are strongly bounded.

(3.3.8) Proposition. *Let R be a Noetherian ring of characteristic p, and let c be a test element for both R and $R[x_1, \dots, x_n]$. If R is strongly bounded relative to c, then $R[x_1, \dots, x_n]$ is strongly bounded relative to c.*

In particular, let R be a local ring such that $R \to \hat{R}$ has regular fibers, and let $d \in R^\circ$ be any element such that $(R_{\text{red}})_d$ is regular. Then d has a power, d^N, which *is a completely stable test element for both R and $R[x_1,\dots,x_n]$. If R is strongly bounded relative to d^N, then so is $R[x_1,\dots,x_n]$.*

Proof. Let ϕ be the R-linear map that makes R strongly bounded relative to c with bounding exponent q. Recall that $(R[x_1,\dots,x_n])^{1/q} \cong R^{1/q}[x_i^{1/q},\dots,x_n^{1/q}]$. We can define a map

$$\theta\colon (R[x_1,\dots,x_n])^{1/q} \to (R[x_1,\dots,x_n])^{1/pq}$$

by $\theta(r^{1/q}) = \phi(r^{1/q})$ and $\theta(x_i^{1/q}) = x_i^{1/q}$. So $\theta(c^{1/q}) = \phi(c^{1/q}) = c^{1/pq}$. To see that θ is $R[x_1,\dots,x_n]$-linear, just note that

$$\theta(\sum r_i x_i) = \sum \theta(r_i)\theta(x_i) = \sum \phi(r_i)x_i = \sum r_i x_i.$$

For the last statement, note that since $(R_{\text{red}})_d$ is regular, some power of d, say d^k, is a test element for R [HH5, Theorem 6.1]. The map $R \to R[x_1,\dots,x_n]$ is smooth, so $R[x_1,\dots,x_n]$ localized at the image of d is also regular. Hence there is a power of the image of d, say $d^{k'}$ which is a test element for $R[x_1,\dots,x_n]$. So d^N is a test element for both rings where $N = \max\{k,k'\}$. □

We now show that a seemingly weaker condition than was necessary in Proposition 3.1.2 is sufficient to have tight closure commute with localization.

(3.3.9) Proposition. *Let R be a complete local domain of characteristic p. Suppose R has the following property:*

(∗) *There exists a finite dimensional vector space, V, of R-forms that are test elements and q_0 such that there is an R^{q_0}-linear map $\theta\colon R \to R$ taking $c^p \to d \in V\backslash\{0\}$ for all $c \in V\backslash\{0\}$.*

Then $W^{-1}I^* \cong (W^{-1}I)^*$ where I is an ideal of R and W is a multiplicative system in R.

Proof. It suffices to see that $(W^{-1}I)^* \subseteq W^{-1}I^*$. Pick $u \in (W^{-1}I)^*$. Set $q = q_0/p$. We have $cu^q \in (W^{-1}I)^{[q]} = W^{-1}I^{[q]}$, so we can pick $f \in W$ such that $fcu^q \in I^{[q]}$. This implies that $f^q cu^q \in I^{[q]}$. So $\mathrm{c}(fu)^q \in I^{[q]}$. Taking pth powers gives $c^p(fu)^{pq} \in I^{[pq]}$. Applying θ gives $d(fu)^{pq} \in I^{[pq]}$. Note that we get a different d for each $Q \geq q_0$. So $I^{[Q]} : fu^Q$ meets $V \backslash \{0\}$ for all $Q \geq q_0$. This means that $d_Q^{1/Q} fu \in IR^\infty$ for all Q. To obtain a notion of order, we shall fix an arbitrary valuation with values in $\mathbb{Z}$, which we denote ord, which is nonnegative on R and positive on m. For example, the usual monomial degree works. We extend the order to R^∞ so that it takes values in $\mathbb{Q}$. Specifically, if $u \in R^{1/q}$, then $\mathrm{ord}(u) = \mathrm{ord}(u^q)/q$. So $\mathrm{ord}(d^{1/Q}) = \mathrm{ord}((d^{1/Q})^Q)/Q = \mathrm{ord}(d)/Q$. Next we claim that V cannot have elements of arbitrarily large order. Let $V_k = \{\, v \in V : \mathrm{ord}(v) > k \,\}$. The V_ks are a decreasing sequence of subspaces, and since V is finite dimensional, they necessarily stabilize. Hence, the elements of V have bounded order. So, $\mathrm{ord}(d_Q^{1/Q}) = \mathrm{ord}(d_Q)/Q \to 0$ as $1/Q \to 0$. This is sufficient to guarantee $fu \in I^*R$ [HH2, Theorem 3.1]. But then $u \in W^{-1}I^*$ as desired. $\square$

We can also show that it is sufficient for the integral closure to have this property.

(3.3.10) Proposition. *Let R be a domain and let S be the integral closure of R in its fraction field. Suppose S has property ($*$) from Proposition 3.3.9. Then R has property ($*$).*

Proof. Pick $b \in R$ such that $bS \subseteq R$. Suppose V, q_0, and θ are the vector space and maps that make S satisfy property $(*)$. Let $V_R = \{\, b^{q_0}v \colon v \in V \,\}$. Define $\tilde{\theta}\colon {V_R}^p \to V$ by

$$\tilde{\theta}((b^{q_0}v)^p) = b^{pq_0}\theta(v^p) = b^{pq_0}d_{q_0}.$$

Then $\tilde{\theta}((b^{q_0}v)^{q_0}) = b^{q_0^2}v^{q_0}$ since $R^{q_0} \subseteq S^{q_0}$ and θ is S^{q_0}-linear. Thus $\tilde{\theta}$ is R^{q_0}-linear. $\square$

CHAPTER 4

CUBICAL CONES

In this chapter we discuss the conjecture that $I^* = IR^+ \cap R$, where R^+ denotes the integral closure of a domain R in an algebraic closure of its fraction field. The ring R^+ is characterized by the property that it is a domain integral over R and every monic polynomial with coefficients in R^+ factors into monic linear factors. This characterization can be used to prove the following property of R^+: If W is a multiplicatively closed set of R, then $(W^{-1}R)^+ \cong W^{-1}R^+$. Aside from providing a much more concrete description of tight closure, proving that $I^* = IR^+ \cap R$ would solve the localization problem for tight closure. It is known that $I^* = IR^+ \cap R$ for parameter ideals [S2] and for rings in which every ideal of the normalization is tightly closed. Also, for those ideals I of an excellent local domain R such that R/I has finite phantom projective dimension, it is known that $I^* = IR^+ \cap R$ [Ab]. However, the conjecture is open even for two-dimensional normal Gorenstein domains. In particular, the conjecture is open for the cubical cone $K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \neq 3$, and more generally for rings of the form $K[[x, y, z]]/(F(x, y, z))$ where F is a homogeneous polynomial.

In this chapter we describe a $\mathbb{Z}_3$-grading on $K[[x, y, z]]/(x^3 + y^3 + z^3)$ which will allow us to show that $I^* = IR^+ \cap R$ in many cases. We will also discuss tight closure and Frobenius closure in these rings before proving the main results, Theorem 4.5.1 and Theorem 6.3.1.

4.1 $\mathbb{Z}_3$-grading

First we describe a $\mathbb{Z}_n$-grading of rings of the form $R = A[[z]]/(z^n - a)$ where $a \in A$. The ring R has the following decomposition as an A-module:

$$R = A + Az + \cdots + Az^{n-1}.$$

This is true because every element of R can be uniquely expressed as an element of $A + Az + \cdots + Az^{n-1}$ by replacing every occurrence of z^n by a. R is $\mathbb{Z}_n$-graded, where the ith piece of R is Az^i, $0 \leq i < n$, since $Az^iAz^j \subseteq Az^{i+j}$ if $i + j < n$ and $Az^iAz^j \subseteq Az^{i+j-n}$ if $i + j \geq n$.

We can use this idea to obtain a $\mathbb{Z}_3$-grading on $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ by letting $A = K[[x, y]]$. Let H, I, and J be ideals of $K[[x, y]]$. Suppose $H \subseteq I \subseteq J \subseteq H\colon (x^3 + y^3)$. Then $H + Iz + Jz^2$ is an ideal of R. On the other hand, in order for a $\mathbb{Z}_3$-graded ideal to be closed under multiplication by z, it must have this form. Thus, it is easy to see that the ideals of R homogeneous with respect to the $\mathbb{Z}_3$-grading are precisely the ideals of this form. We can study the ideal $H + Iz + Jz^2$ by considering (H, I, J), a triple of ideals in $K[[x, y]]$. Indeed, we will use the notation (H, I, J) to denote the ideal $H + Iz + Jz^2$, and it is understood that H, I, and J are ideals of $K[[x, y]]$. One of the advantages of looking at triples of ideals in $K[[x, y]]$ instead of ideals in $K[[x, y, z]]/(x^3 + y^3 + z^3)$ is that $K[[x, y]]$ is a regular ring, and the Frobenius endomorphism is flat on regular rings, among other things.

Next we observe that the $\mathbb{Z}_3$-grading on R extends to R^∞. It is enough to show that the grading on R extends to $R^{1/q}$. If $u \in R_i$, then the image of u is in $R_j^{1/q}$ where $qi \equiv j \mod 3$.

We now show that if I is a graded ideal, then so is I^*. First note that R is invariant under the following transformations:

$$\begin{aligned} x &\mapsto x & \qquad & & x &\mapsto x \\ y &\mapsto y & \qquad & \text{and} & y &\mapsto y \\ z &\mapsto \alpha z & & & z &\mapsto \beta z \end{aligned}$$

where α and β are cube roots of unity.

In general, if M is a $\mathbb{Z}_n$-graded R-module and R contains all of the nth roots of unity, then we have the following automorphisms: $\phi_j \colon m_i \mapsto \alpha_j^i m_i$ where $\{\alpha_j\}$ is the set of nth roots of unity. We first note that any submodule stable under these maps is graded.

(4.1.1) Proposition. *Let M be a $\mathbb{Z}_n$-graded module and $W \subseteq M$ a submodule. Suppose $\phi_j(W) \subseteq W$ where ϕ_j are the automorphisms of M that map $m_i \mapsto \alpha_j^i m_i$ where $\{\alpha_j\}$ is the set of nth roots of unity. Then W is a graded submodule.*

Proof. Pick $w_0 + \cdots + w_{n-1} \in W$. Then we know that $\phi_j(w_0 + \cdots + w_{n-1}) = w_0 + \cdots + \alpha^{n-1} w_{n-1} \in W$. Writing this in matrix form gives:

$$\begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{n-1} \\ \vdots & & \ddots & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{n-1} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} w'_0 \\ w'_1 \\ \vdots \\ w'_{n-1} \end{pmatrix}$$

or $A\bar{w} = \bar{w}'$ where $w_j, w'{}_j \in W$. But A is just a Vandermonde matrix, so $\det A = \prod_{0 \le i < j \le n} (\alpha_i - \alpha_j)$. This determinant is nonzero as long as all the roots of unity are in K. To see that $w_i \in W$ it is enough to write w_i as a linear combination of elements of W. We can write w_i as a linear combination of w'_j by using the inverse of A. $\square$

Next we note that if an ideal I is invariant under the automorphisms mentioned in (4.1.1), then I^* is graded.

(4.1.2) Lemma. *Let R be a $\mathbb{Z}_n$-graded ring and I an ideal of R. Suppose $\phi_j(I) \subseteq I$ where ϕ_j are the automorphisms of R that map $r_i \mapsto \alpha_j^i r_i$ where $\{\alpha_j\}$ is the set of nth roots of unity. Then I^* is a graded ideal of R.*

Proof. Let $u \in I^*$. Then $\phi(u) \in \phi(I)^*$. Since I is stable under ϕ, we know that $\phi(u) \in I^*$. In other words, I^* is stable under ϕ and therefore I^* is graded by Proposition 4.1.1. $\square$

4.2 Frobenius Closure

In order to show that $I^* = IR^+ \cap R$ for certain classes of ideals, we will make use of another closure operation, Frobenius closure. Recall that $I^F = \{u \in R : u^q \in I^{[q]} \text{ for some } q\}$. We can also think of I^F as $IR^\infty \cap R$, so $I^F \subseteq IR^+ \cap R$, since $R^\infty \subseteq R^+$. In addition, we know that $IR^+ \cap R \subseteq I^*$ [HH6]. Hence

$$I^F \subseteq IR^+ \cap R \subseteq I^*.$$

So, if $I^F = I^*$, then that implies that $I^* = IR^+ \cap R$.

There is an interesting bifurcation of this question in $R = K[[x, y, z]]/(x^3+y^3+z^3)$ depending on the characteristic of K. If K has characteristic p and $p \equiv 1 \mod 3$, then R is F-pure (Lemma 4.3.2) and $I^F = I$. We know that $I^* \neq I$ for some ideals of R, so I^F cannot equal I^*, although it is still possible that $I^* = IR^+ \cap R$. If $p \equiv 2 \mod 3$, then R is not F-pure and it is conjectured that $I^F = I^*$ and hence that $I^* = IR^+ \cap R$.

The goal of this chapter is to show that $I^* = I^F$, and hence $I^* = IR^+ \cap R$, for many graded ideals of R when the characteristic of K is congruent to 2 mod 3.

4.3 Test Elements in Cubical Cones

We begin by establishing some useful results about test elements and some technical lemmas. The following proposition is proved for char $K \neq 2, 3$ using a somewhat different method in [S3].

(4.3.1) Proposition. *Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \neq 3$. Then the maximal ideal, m, is the test ideal.*

Proof. First note that we can reduce to the case where K is algebraically closed. Enlarging K to an algebraic closure is an integral extension and will not affect tight closure.

Let τ be the parameter test ideal for R. By Proposition 4.4(iii) of [S3], we know that $\tau = \{\, c \in R \text{ such that } c(x^t, y^t)^* \subset (x^t, y^t) \text{ all } t \in \mathbb{N} \,\}$. Since R is Gorenstein, the test ideal is the same as the parameter test ideal [S3, Proposition 4.4].

We will show that $(x^t, y^t)^* = (x^t, y^t, x^{t-1}y^{t-1}z^2)$. Then it is clear that $\tau = (x, y, z)$ since $(x^t, y^t) \colon (x^t, y^t, x^{t-1}y^{t-1}z^2) = (x, y, z)$. Let $I = (x^t, y^t)$ and $J = (x^t, y^t, x^{t-1}y^{t-1}z^2)$. The socle mod J is generated by $u_1 = x^{t-2}y^{t-1}z^2$, $u_2 = x^{t-1}y^{t-2}z^2$ and $u_3 = x^{t-1}y^{t-1}z$. To see that $I^* = J$, it suffices to show that $\sum K u_i \cap I^* = 0$, since if $J \subsetneq I^*$, then I^* has nonzero intersection with $J \colon m$.

We would like to see that

$$\lambda_1 x^{t-2}y^{t-1}z^2 + \lambda_2 x^{t-1}y^{t-2}z^2 + \lambda_3 x^{t-1}y^{t-1}z \notin (x^t, y^t)^*.$$

Using the $\mathbb{Z}_3$-grading, it is enough to check that $\lambda_3 x^{t-1}y^{t-1}z \notin (x^t, y^t)^*$ and $\lambda_1 x^{t-2}y^{t-1}z^2 + \lambda_2 x^{t-1}y^{t-2}z^2 \notin (x^t, y^t)^*$. Using the $\mathbb{Z}_3$-grading again, but now letting x play the role of z ($R = A[[x]]/(x^3 - a)$, $A = K[[y, z]]$), we can reduce the problem to checking $\lambda_1 x^{t-2}y^{t-1}z^2 \notin (x^t, y^t)^*$, $\lambda_2 x^{t-1}y^{t-2}z^2 \notin (x^t, y^t)^*$ and $\lambda_3 x^{t-1}y^{t-1}z \notin (x^t, y^t)^*$.

Suppose $x^{t-1}y^{t-1}z \in (x^t, y^t)^*$. Then $z \in (x^t, y^t)^* \colon x^{t-1}y^{t-1}$. We claim that $(x^t, y^t)^*\colon$ $x^{t-1}y^{t-1} \subseteq (x, y)^*$. Let $u \in (x^t, y^t)^* \colon x^{t-1}y^{t-1}$, so $ux^{t-1}y^{t-1} \in (x^t, y^t)^*$. Then there exists c such that $cu^q x^{(t-1)q}y^{(t-1)q} \in (x^{tq}, y^{tq})$. This implies that $cu^q \in (x^{tq}, y^{tq}) \colon x^{(t-1)q}y^{(t-1)q}$. But $(x^{tq}, y^{tq}) \colon x^{(t-1)q}y^{(t-1)q} \subseteq (x^q, y^q)^*$ by a colon capturing argument [HH1, Theorem 7.15a]. So $cu^q \in (x^q, y^q)^*$, and we can find a test element d such that $cdu^q \in (x^q, y^q)$ for all q. In other words, $u \in (x, y)^*$. Thus $x^{t-1}y^{t-1}z \in (x^t, y^t)^*$ implies $z \in (x, y)^*$, but we know that $z \notin (x, y)^*$ by a degree argument [S4, Theorem 2.2].

Now suppose $x^{t-2}y^{t-1}z^2 \in (x^t, y^t)^*$. This implies that $z^2 \in (x^t, y^t)^* \colon x^{t-2}y^{t-1}$. Using the same argument as before, we can show that $(x^t, y^t)^* \colon x^{t-2}y^{t-1} \subseteq (x^2, y)^*$. By symmetry, we must also have $z^2 \in (x, y^2)^*$. So $z^2 \in (x^2, y)^* \cap (x, y^2)^*$ which is contained in $(x^2, xy, y^2)^*$ by Theorem 7.12 of [HH1]. Again, $z^2 \notin (x^2, xy, y^2)^*$ by degree arguments [S4, Theorem 2.2]. $\square$

There is also a simple proof that $\tau = m$ in $K[[x, y, z]]/(x^3 + y^3 + z^3)$ if K is a field of characteristic p and $p \equiv 1 \mod 3$.

First we prove the following lemma:

(4.3.2) Lemma. *Let* $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$. *Let* K *be a field of characteristic* p *with* $p \equiv 1 \mod 3$. *Then* R *is F-pure.*

Proof. R is F-pure if and only if the obvious map $E \to E \otimes R^{1/p}$ is injective where $E = E(R/m)$ [HR]. Since $\operatorname{Soc} E \hookrightarrow E$ is an essential extension, $E \to E \otimes R^{1/p}$ is injective if and only if the socle generator gets mapped to a non-zero element in $E \otimes R^{1/p}$. Here

$$E(R/m) = \varinjlim \frac{R}{(x^t, y^t)}.$$

A generator of the socle is $[z^2 + (x, y)]$. Thus R is not F-pure if and only if $z^2 \in (x, y)R^{1/p} \cap R$. This is true if and only if $(z^2)^p \in (x^p, y^p)$. If $p \equiv 1 \mod 3$, say $p = 3h + 1$, then $2p = 6h + 2$. Using the basic relation in R we see that

$$(z^{2p}) = (z^3)^{2h} z^2 = (x^3 + y^3)^{2h} z^2 = \sum_{i=0}^{2h} \tbinom{2h}{i} x^{3(2h-i)} y^{3i} z^2.$$

Since $2h < p$, $\binom{2h}{h} x^{3h} y^{3h} z^2 \not\equiv 0 \mod p$ and also is not contained in (x^p, y^p). So $(z^2)^p \notin (x^p, y^p)$ and R is F-pure in this case. $\square$

(4.3.3) Lemma. *Let* R *be a Noetherian ring of characteristic* p. *If* R *is F-pure, then the test ideal is radical.*

Proof. This is Proposition 2.5 of [FeW]. Suppose c^{q_0} is a test element. $u \in I^*$ implies that $c^{q_0} u^{qq_0} \in I^{[qq_0]}$ for all q. So $(cu^q)^{q_0} \in (I^{[q]})^{[q_0]}$ for all q. Since R is F-pure, $I = I^F$ for any ideal I of R where I^F denotes the Frobenius closure of I. Thus $cu^q \in I^{[q]}$ for all q, and so c is a test element. $\square$

(4.3.4) Proposition. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$ with K a field of characteristic p, and let m be the maximal ideal in R. Suppose $p \equiv 1 \mod 3$. Then m is the test ideal.*

Proof. This Corollary 2.6 of [FeW]. If R_c is regular, then c^n is a test element for R for some n [HH5]. Since R is an isolated singularity, R_c is regular for any element $c \in m$. So $m^N \subseteq \tau$ for some N. We also know by the previous lemma that if R is F-pure, then the test ideal is radical. Since $m^N \subseteq \tau$ and τ is radical, we conclude that $m \subseteq \tau$. Since R is not F-regular, the test ideal is a proper ideal and so $m = \tau$. $\square$

The fact that m is the test ideal provides quite a lot of information. For example, using the fact that m is the test ideal, we may conclude that if $u \in I^*\backslash I$, then u is in the socle mod I.

(4.3.5) Proposition. *Let (R,m) be a local ring. Suppose m is the test ideal. If $u \in I^*\backslash I$, then u is in the socle mod I.*

Proof. Let $u \in I^*\backslash I$. Then $mu^q \subseteq I^{[q]}$ for all q. In particular, $mu \subseteq I$. This says exactly that u is in the socle mod I. $\square$

(4.3.6) Remark. Although determining whether an element is in the tight closure or Frobenius closure of an ideal involves checking certain conditions for infinitely many values of $q = p^e$, there are some instances where one q is enough. If c is a test element and $cu^q \notin I^{[q]}$ for some q, then $u \notin I^*$. Similarly, if $u^q \in I^{[q]}$ for some q, then $u^{q'} \in I^{[q']}$ for all $q' \geq q$ and hence $u \in I^{\mathrm{F}}$.

In either situation, since we only need one q that works, we can pick whichever value of q is most helpful. For example, when $p \equiv 2 \mod 3$, $p^{2e} \equiv 1 \mod 3$ and $p^{2e+1} \equiv 2 \mod 3$. It is often easier to work with powers of p with a particular residue mod 3 and so we may choose q accordingly. In Chapters 4 and 6 we routinely pick whichever element of the test ideal, m, is most useful as a test element and whichever value of q is most helpful.

4.4 Applications of the $\mathbb{Z}_3$-grading to Tight Closure

Next we will show how to use the $\mathbb{Z}_3$-grading to answer questions about tight closure and Frobenius closure. When trying to determine I^* and I^F for a given ideal I, we are interested in calculating $I^{[q]}$ and $I\colon m$. We will first calculate $I\colon m$.

(4.4.1) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, and let $H + Iz + Jz^2$ be a $\mathbb{Z}_3$-graded ideal in R. Then $(H+Iz+Jz^2)\colon(x,y,z) =$*

$$\big((H\colon(x,y)) \cap I\big) + \big((I\colon(x,y)) \cap J\big)\ z + \big((J\colon(x,y)) \cap (H\colon(x^3+y^3))\big)\ z^2.$$

Proof. Write $R = R_0 + R_1 + R_2$. Suppose $r \in R_0$ and $r \in (H+Iz+Jz^2)\colon(x,y,z)$. So we must have $r(x,y) \subseteq H$ and $rz \in Iz$. In other words, $r \in \big(H\colon(x,y)\big) \cap I$. Similarly, if $rz \in R_1$ and $rz \in (H+Iz+Jz^2)\colon(x,y,z)$, we must have $r \in \big(I\colon(x,y)\big) \cap J$. Let $rz^2 \in R_2$ and suppose $r \in (H+Iz+Jz^2)\colon(x,y,z)$. Again, we see that $r \in J\colon(x,y)$. We also know that $(rz^2)z = r(x^3+y^3) \in (H+Iz+Jz^2)$. Since $r(x^3+y^3) \in R_0$, we must have $r(x^3+y^3) \in H$. In other words, $r \in H\colon(x^3+y^3)$. So $r \in \big((J\colon(x,y)) \cap ((H\colon(x^3+y^3))\big)$. $\square$

Next we will determine $I^{[q]}$ when $q \equiv 2 \mod 3$.

(4.4.2) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let $q = p^{2e+1} = 3h+2$ and let $f = x^3+y^3$. Let*

$H + Iz + Jz^2$ be a $\mathbb{Z}_3$-graded ideal in R. Then

$$(H + Iz + Jz^2)^{[q]} = (H^{[q]} + I^{[q]} f^{h+1} + J^{[q]} f^{2h+2})$$
$$+ (H^{[q]} + I^{[q]} f^{h+1} + J^{[q]} f^{2h+1})z + (H^{[q]} + I^{[q]} f^{h} + J^{[q]} f^{2h+1})z^2.$$

Let $u = u_0 + u_1 z + u_2 z^2$. Then the question of whether $u^q \in (H + Iz + Jz^2)^{[q]}$ in R reduces to the following three inclusions in $K[[x, y]]$:

$$u_0{}^q \in (H^{[q]} + I^{[q]} f^{h+1} + J^{[q]} f^{2h+2})$$

$$u_1{}^q f^h \in (H^{[q]} + I^{[q]} f^{h} + J^{[q]} f^{2h+1})$$

$$u_2{}^q f^{2h+1} \in (H^{[q]} + I^{[q]} f^{h+1} + J^{[q]} f^{2h+1}).$$

Proof. We start by noting that $(H+Iz+Jz^2)^{[q]}$ is generated by $H^{[q]}+I^{[q]}z^q+J^{[q]}z^{2q}$. Rewriting this using $q = 3h+2$ and the basic relation in R, $z^3 = -(x^3+y^3)$, yields $H^{[q]} + I^{[q]} f^h z^2 + J^{[q]} f^{2h+1} z$. We will first consider $(H + Iz + Jz^2)^{[q]} \cap R_0$. If we multiply $I^{[q]} f^h z^2$ by z, we get $I^{[q]} f^h z^3 = I^{[q]} f^{h+1}$ which is in R_0. Similarly, multiplying $J^{[q]} f^{2h+1} z$ by z^2 gives $J^{[q]} f^{2h+1} z^3 = J^{[q]} f^{2h+2}$. Thus,

$$(H + Iz + Jz^2)^{[q]} \cap R_0 = (H^{[q]} + I^{[q]} f^{h+1} + J^{[q]} f^{2h+2}).$$

Similar arguments show that

$$(H + Iz + Jz^2)^{[q]} \cap R_1 = (H^{[q]} + I^{[q]} f^{h} + J^{[q]} f^{2h+1})$$

and

$$(H + Iz + Jz^2)^{[q]} \cap R_2 = (H^{[q]} + I^{[q]} f^{h+1} + J^{[q]} f^{2h+1}).$$

The last statement in the lemma is now clear. □

Next we determine $I^{[q]}$ when $q \equiv 1 \mod 3$.

(4.4.3) Lemma. *Let* $R = K[[x,y,z]]/(x^3+y^3+z^3)$, *where* K *is a field of characteristic* p *and* $p \equiv 2 \mod 3$. *Let* $q = p^{2e} = 3h+1$ *and let* $f = x^3+y^3$. *Let* $H + Iz + Jz^2$ *be a* $\mathbb{Z}_3$*-graded ideal in* R. *Then*

$$(H+Iz+Jz^2)^{[q]} = (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1}) \\ + (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1})z + (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h})z^2.$$

Let $u = u_0 + u_1z + u_2z^2$. *Then the question of whether* $u^q \in (H+Iz+Jz^2)^{[q]}$ *in* R *reduces to the following three inclusions in* $K[[x,y]]$:

$$u_0{}^q \in (H^{[q]} + I^{[q]}f^{h+1} + J^{[q]}f^{2h+1})$$
$$u_1{}^q f^h \in (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h+1})$$
$$u_2{}^q f^{2h} \in (H^{[q]} + I^{[q]}f^{h} + J^{[q]}f^{2h})$$

Proof. The proof is identical to the proof of Lemma 4.4.2 except we use $q = 3h+1$. □

Using the above information we can see that $f^h = (x^3+y^3)^h$ appears often in the inclusions. It turns out that the question of whether a given element is in the tight closure of an ideal often comes down to whether or not a certain power of f is contained in (x^q, y^q). To this end, we establish the following lemmas which will be useful in showing that $I^* = I^F$.

(4.4.4) Lemma. *Let* $A = K[[x,y]]$, *where* K *is a field of characteristic* p *and* $p \equiv 2 \mod 3$. *Let* $p = 3h+2$ *and let* $f = x^3+y^3$. *Then* $f^{2h} \notin (x^p, y^p)$. *Let* $q = p^{2e} = 3k+1$, *then* $f^{2k} \in (x^q, y^q)$.

Proof. Expanding f^{2h} using the binomial theorem gives

$$f^{2h} = (x^3 + y^3)^{2h} = sum_{i=0}^{2h} \binom{2h}{i} x^{3(2h-i)} y^{3i}.$$

Since $\binom{2h}{h} x^{3h} y^{3h}$ is a term in the expansion and $x^{3h}y^{3h} \notin (x^{3h+2}, y^{3h+2}) = (x^p, y^p)$, it suffices to see that $\binom{2h}{h} \not\equiv 0 \mod p$. But $2h < p$, so p does not divide $\binom{2h}{h}$.

As in the above case, $f^{2k} \in (x^q, y^q)$ if and only if $\binom{2k}{k} \equiv 0 \mod p$. To prove the second claim, we will make use of a Frobenius closure argument in $K[[x, y, z]]/(x^3 + y^3 + z^3)$. Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$. Suppose we know that $z^{2q} \in (x^q, y^q)R$ where $q = 3k + 1$. Using the basic relation in R we see that

$$z^{6k+2} = z^{6k}z^2 = (x^3 + y^3)^{2k} z^2 = (\cdots + \tbinom{2k}{k} x^{3k} y^{3k} + \cdots)z^2$$

So $z^{2q} \in (x^q, y^q)R$ if and only if $(\cdots + \binom{2k}{k} x^{3k}y^{3k} + \cdots)z^2 \in (x^q, y^q)R$. Using the $\mathbb{Z}_3$-grading we see that this is equivalent to having $(\cdots + \binom{2k}{k} x^{3k}y^{3k} + \cdots) \in (x^q, y^q)A$. This, in turn, is equivalent to having $\binom{2k}{k} \equiv 0 \mod p$. In other words, $\binom{2k}{k} \equiv 0 \mod p$ if and only if $z^{2q} \in (x^q, y^q)R$ where $q = 3k + 1$. We know that $z^{2p} \in (x^p, y^p)R$ when $p \equiv 2 \mod 3$ by the proof of Proposition 6.1.3. This implies that $z^{2q} \in (x^q, y^q)$ for all $q = p^e$. In particular, $z^{2q} \in (x^q, y^q)$ when $q = 3k + 1$. So we have $z^{2q} \in (x^q, y^q)$ when $q = 3k + 1$ which implies $\binom{2k}{k} \equiv 0 \mod p$, and this in turn implies $f^{2k} \in (x^q, y^q)$. $\square$

We will use the following result about calculating binomial coefficients mod p in Lemma 4.4.6.

(4.4.5) Lucas's Theorem. *Let p be a prime and let $n = \sum_0^s a_i p^i$, $0 \leq a_j < p$,*

$m = \sum_0^s b_i p_i$, $0 \leq b_k < p$. *Then*

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_s}{b_s} \mod p.$$

Proof. See [Fi, Theorem 1] or [L, p230]. $\square$

(4.4.6) Lemma. *Let* $A = K[[x,y]]$*, where* K *be a field of characteristic* p *and* $p \equiv 2 \mod 3$*. Let* $q = p^{2e} = 3h+1$ *and* $f = x^3 + y^3$*. Then* $\binom{3h-2}{h-1} \not\equiv 0 \mod p$ *and* $f^{2h-2} \in (x^q, y^q)$ *except when* $q = 25$*.*

Proof. Since $p^{2e} = 3h+1$, we can write $3h-2 = p^{2e}-3$. So

$$\binom{3h-2}{h-1} = \frac{(p^{2e}-3)(p^{2e}-4)\cdots(p^{2e}-(h+1))}{1\cdot 2\cdots(h-1)}.$$

We know that p^k divides $(p^{2e}-n)$ if and only if p^k divides n for $k \leq 2e$. If we match up terms in the numerator and denominator, it is clear that

$$\frac{1}{1\cdot 2}\cdot\frac{(p^{2e}-3)(p^{2e}-4)\cdots(p^{2e}-(h-1))}{3\cdots(h-1)}\cdot\frac{(p^{2e}-h)(p^{2e}-(h+1))}{1}$$

is divisible by p if and only if $((p^{2e}-h)(p^{2e}-(h+1))/2$ is divisible by p.

If $p \neq 2$, then $((p^{2e}-h)(p^{2e}-(h+1))/2$ is divisible by p if and only if p divides $h(h+1)$. Suppose p divides h, say $h = ph_0$. Then $p^{2e} = 3ph_0 + 1$, which implies that p divides 1. Similarly, if p divides $(h+1)$, say $h+1 = ph_0$. Then $p^{2e} = 3(ph_0 - 1) + 1 = 3ph_0 - 2$, which implies p divides 2.

If $p = 2$, then we have 2 divides $\binom{3h-2}{h-1}$ if and only if 2 divides $(p^{2e}-h)(p^{2e}-(h+1))/2$. This is true if and only if 4 divides $(p^{2e}-h)(p^{2e}-(h+1))$. If 4 divides $2^{2e}-h$, then 4 divides h, or $h \equiv 0 \mod 4$. Similarly, if 4 divides $2^{2e}-(h+1)$, then

4 divides $(h+1)$ which implies that $h \equiv -1 \mod 4$. However, $2^{2e} = 3h+1$ implies that $h \equiv 1 \mod 4$.

To see that $f^{2h-2} \in (x^q, y^q)$, we expand f^{2h-2} using the binomial theorem. This gives

$$f^{2h-2} =$$

$$(\cdots + \tbinom{2h-2}{h-3} x^{3h+3} y^{3h-9} + \tbinom{2h-2}{h-2} x^{3h} y^{3h-6} + \tbinom{2h-2}{h-1} x^{3h-3} y^{3h-3}$$
$$+ \tbinom{2h-2}{h} x^{3h-6} y^{3h} + \tbinom{2h-2}{h+1} x^{3h-9} y^{3h+3} + \cdots).$$

Only the middle three terms pose any potential problem, so it is sufficient to show that $\binom{2h-2}{h-2}$, $\binom{2h-2}{h-1}$ and $\binom{2h-2}{h}$ are all congruent to zero mod p. Since $\binom{2h-2}{h} = \binom{2h-2}{h-2}$, we need only consider $\binom{2h-2}{h}$ and $\binom{2h-2}{h-1}$.

To see that $\binom{2h-2}{h} \equiv 0 \mod p$ and $\binom{2h-2}{h-1} \equiv 0 \mod p$, first note that $\binom{2h-2}{h} = \frac{(h-1)}{h}\binom{2h-2}{h-1}$. We know that p does not divide h as that would imply that p divides 1. Similarly, if p divides $h-1$, then p divides 4. So if $p \neq 2$, then p divides $\binom{2h-2}{h}$ if and only if p divides $\binom{2h-2}{h-1}$.

We can also show that $\binom{2h}{h} = \frac{2(2h-1)}{h}\binom{2h-2}{h-1}$. If $p \neq 2$, we know that p does not divide h, and if p divides $2h-1$, then p divides 5. So if $p \neq 2, 5$, then p divides $\binom{2h}{h}$ if and only if p divides $\binom{2h-2}{h-1}$.

We know from Proposition 4.4.4 that p divides $\binom{2h}{h}$ for the values of h we are considering, so if $p \neq 2, 5$, we know that p also divides $\binom{2h-2}{h-1}$ and $\binom{2h-2}{h}$.

If $p = 2$, using Lucas' Theorem (4.4.5) to compute binomial coefficients in characteristic p, we can show that

$$\binom{\text{even}}{\text{odd}} \equiv 0 \mod 2.$$

Since $q = 3h + 1$ is even, h must be odd. This implies that $2h - 2$ is even. Thus

$$\binom{2h-2}{h} = \binom{\text{even}}{\text{odd}} \equiv 0 \mod 2.$$

It is also known that

$$\binom{2k}{k} \equiv 0 \mod 2 \text{ for any } k.$$

Using this we see that

$$\binom{2h-2}{h-1} = \binom{2(h-1)}{h-1} \equiv 0 \mod 2.$$

It remains to see that $\binom{2h-2}{h} \equiv 0 \mod 5$ and $\binom{2h-2}{h-1} \equiv 0 \mod 5$. We know that if $p \neq 2$, then p divides $\binom{2h-2}{h}$ if and only if p divides $\binom{2h-2}{h-1}$, so it is enough to show that $\binom{2h-2}{h} \equiv 0 \mod 5$. Write $5^{2e} = 3h + 1$. We claim that h has the following 5-ary expansion:

$$h = 5^{2e-1} + 3 \cdot 5^{2e-2} + 5^{2e-3} + 3 \cdot 5^{2e-4} + \cdots + 5 + 3 \cdot 5^0.$$

In other words, $h = \sum_{i=0}^{2e-1} a_i 5^i$ where $a_i = 1$ if i is odd and $a_i = 3$ if i is even. To see this, write $h = \sum a_i p^i$. We are assuming that $3h + 1 = 3(\sum a_i p^i) + 1 = 5^{2e}$. This implies that $3a_0 + 1 \equiv 0 \mod 5$ with $0 \leq a_0 < 5$. Hence we must have $a_0 = 3$ and we will carry a 2 to the next place. Thus $3a_1 + 2 \equiv 0 \mod 5$ with $0 \leq a_1 < 5$. Hence $a_1 = 1$ and we will carry a 1 to the next term. So $3a_2 + 1 \equiv 0 \mod 5$ and the pattern continues. In other words, $3a_i + 1 \equiv 0 \mod 5$ if i is even and $3a_i + 2 \equiv 0 \mod 5$ if i is odd. Since $h = 5^{2e-1} + 3 \cdot 5^{2e-2} + 5^{2e-3} + 3 \cdot 5^{2e-4} + \cdots + 5 + 3 \cdot 5^0$, we can compute $2h - 2 = 3 \cdot 5^{2e-1} + 5^{2e-2} + \cdots + 2 \cdot 5 + 4 \cdot 5^0$. Using Lucas' Theorem, we see that

$$\binom{2h-2}{h} \equiv \binom{3}{1}\binom{1}{3} \cdots \binom{3}{1}\binom{1}{3}\binom{2}{1}\binom{4}{3} \equiv 0 \mod 5$$

as long as $5^{2e} > 25$. $\square$

At times, we will be able to make use of the fact that we are working over a regular ring or that $K[[x,y,z]]/(x^3+y^3+z^3)$ is flat as a $K[[x,y]]$-module. The following lemma and corollary provide useful information in these situations.

(4.4.7) Lemma. *Let R, S be arbitrary Noetherian rings such that S is a flat R-algebra, and let I, J be ideals of R. Then $IS{:}_S JS = (I{:}_R J)S$, where $I{:}_R J = \{\, r \in R : rJ \subseteq I \,\}$.*

Proof. See [N, Theorem 18.1, part 2].

(4.4.8) Corollary. *In a regular ring R of characteristic p, for any two ideals I, J we have $I^{[q]} :_R J^{[q]} = (I :_R J)^{[q]}$ for all q. In particular, $I^{[q]}{:}\, x^q = (I{:}\, x)^{[q]}$ for all q.*

Proof. See Corollary 4.3 of [HH1]. The statement follows from Lemma 4.4.7, since the iterated Frobenius endomorphism F^e: $R \to R$ is flat when R is regular [K, Theorem 2.1] and $I^{[q]} = F^e(I)R$.

4.5 Tight Closure and Frobenius Closure in Cubical Cones

We can now show that $I^* = I^F$ for some not necessarily irreducible ideals. We will discuss irreducible ideals in Chapter 6.

(4.5.1) Proposition. *Let I be a $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let $f = x^3+y^3$. If I has any of the following forms, then $I^* = I^F$.*

(1) (H, H, H)

(2) $\big(H, H, H{:}\,(x,y)\big)$

(3) $\big(H, H\colon (x,y), H\colon f\big)$

(4) $\big(H, H\colon (x,y), H\colon (x,y)\big)$

(5) $\big(H, H, H\colon (x^2,y)\big)$

In fact, in (2)-(5), I is tightly closed, i.e. $I = I^*$.

Proof. We know that if $u \in I^* \backslash I$, then u is in the socle mod I (Proposition 4.3.5), so it is sufficient to check whether elements of the socle are in I^* and I^F.

Proof of (1). Let $q = 3h+2$. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.1) we know that $I\colon (x,y,z) = H + Hz + \big(H\colon (x,y)\big)z^2$. So the socle mod I is in R_2, the second graded piece of R. Let $u \in \big(H\colon (x,y)\big)\backslash H$. Then uz^2 represents an element of the socle in R_2. We know that the test ideal is (x,y,z) by Proposition 4.3.1. If $uz^2 \in I^*$, then, using z as a test element, we must have $zu^q z^{2q} \in I^{[q]}$ for all q. Using the grading (Lemma 4.4.2), we see that this is equivalent to having

$$u^q f^{2h+1} \in H^{[q]} + H^{[q]} f^h + H^{[q]} f^{2h+1}$$

in $K[[x,y]]$, which implies that

$$u^q f^{2h+1} \in H^{[q]}.$$

This, however, is exactly what is needed to have $(uz^2)^q \in I^{[q]}$ (Lemma 4.4.2) and hence $uz^2 \in I^F$.

We can also show that $I^* \neq I$ in this case, in other words, uz^2 is always in I^*. In fact we can show show that $uz^2 \in I^F$. If $uz^2 \in I^F$, then we must have

$$u^q z^{2q} \in I^{[q]}.$$

This is equivalent to having

$$z^{2q} \in I^{[q]} :_R u^q.$$

Since R is a flat $K[[x,y]]$-algebra,

$$I^{[q]} :_R u^q = (I^{[q]} :_{K[[x,y]]} u^q)R$$

(Lemma 4.4.7). Since $K[[x,y]]$ is a regular local ring, we may move the qth powers outside of the colon to get

$$(I^{[q]} :_{K[[x,y]]} u^q)R = (I :_{K[[x,y]]} u)^{[q]}R$$

(Corollary 4.4.8). Since I is just the expansion of H to R,

$$(I :_{K[[x,y]]} u)^{[q]}R = (H :_{K[[x,y]]} u)^{[q]}R = (x,y)^{[q]}R = (x^q, y^q)R.$$

Thus, $uz^2 \in I^F$ if and only if $z^{2q} \in (x^q, y^q)$, which it is by the proof of Proposition 6.1.3.

Proof of (2). In this case $I = \big(H, H, H : (x,y)\big)$. Let q be an even power of p. Then $q \equiv 1 \mod 3$, so let $q = 3h+1$. Using the $\mathbb{Z}_3$-grading we know that $I : (x,y,z) = H + \big(H : (x,y)\big)z + \big(H : (x^2, xy, y^2)\big)z^2$ (Lemma 4.4.1). So the socle has components in R_1 and R_2. Let $u \in \big(H : (x,y)\big) \backslash H$. Then uz represents an element of the socle mod I and uz is in R_1. If $uz \in I^*$, then, using x as a test element, we must have

$$xu^q f^h \in H^{[q]} + H^{[q]} f^h + \big(H : (x,y)\big)^{[q]} f^{2h+1}$$

in $K[[x,y]]$ (Lemma 4.4.3). Using the fact that we are working over a regular ring, we can write this as

$$xu^q f^h \in H^{[q]} + H^{[q]} f^h + \big(H^{[q]} : (x,y)^{[q]}\big) f^{2h+1}$$

(Corollary 4.4.8), or

$$xu^q f^h \in H^{[q]} + H^{[q]} f^h + \bigl(H^{[q]} \colon (x^q, y^q)\bigr) f^{2h+1}.$$

Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 4.4.4), we know that

$$xu^q f^h \in H^{[q]} + H^{[q]} f^h + H^{[q]},$$

which certainly implies that $xu^q f^h \in H^{[q]}$. This, in turn, implies that $xf^h \in H^{[q]} \colon u^q$. Again, using the fact that we are working over a regular ring, we see that $xf^h \in (H \colon u)^{[q]}$ (Corollary 4.4.8). Now $(x, y) \subseteq H \colon u$, and since $u \notin H$, $H \colon u$ must be a proper ideal. In other words, $H \colon u = (x, y)$. So now we have that $xf^h \in (x, y)^{[q]} = (x^q, y^q)$. But $xf^h \notin (x^q, y^q)$. To see this just expand $f^h = x^{3h} + \cdots + y^{3h}$ and see that $xf^h = (\cdots + xy^{3h}) \notin (x^{3h+1}, y^{3h+1})$. Thus $uz \notin I^*$.

Let $u \in \bigl(H \colon (x^2, xy, y^2)\bigr) \backslash \bigl(H \colon (x, y)\bigr)$. Then uz^2 represents an element of the socle mod I and uz^2 is in R_2. If $uz^2 \in I^*$, then, using x as a test element, we must have

$$xu^q f^{2h} \in H^{[q]} + H^{[q]} f^h + \bigl(H \colon (x, y)\bigr)^{[q]} f^{2h}$$

in $K[[x, y]]$ (Lemma 4.4.3). Using the fact that we are working over a regular ring, we can write this as

$$xu^q f^{2h} \in H^{[q]} + H^{[q]} f^h + \bigl(H^{[q]} \colon (x, y)^{[q]}\bigr) f^{2h}$$

(Corollary 4.4.8), or

$$xu^q f^{2h} \in H^{[q]} + H^{[q]} f^h + \bigl(H^{[q]} \colon (x^q, y^q)\bigr) f^{2h}.$$

Since $f^{2h} \in (x^q, y^q)$ (Lemma 4.4.4), we know that

$$xu^q f^{2h} \in H^{[q]} + H^{[q]} f^h + H^{[q]},$$

which certainly implies that $xu^q f^{2h} \in H^{[q]}$. This, in turn, implies that $xf^{2h} \in H^{[q]}\colon u^q$. Again, using the fact that we are working over a regular ring, we see that $xf^{2h} \in (H\colon u)^{[q]}$ (Corollary 4.4.8). Now $(x^2, xy, y^2) \subseteq H\colon u$, and since $u \notin H\colon (x, y)$, $H\colon u \neq (x, y)$. In other words, $(x^2, xy, y^2) \subseteq H\colon u \subsetneq (x, y)$. Since $K[x, y]/(x^2, xy, y^2) \cong K + Kx + Ky$, we know that $H\colon u = (x^2, xy, y^2)$ or (x, y^2) or (x^2, y) or $(x^2, xy, y^2, x + \lambda y)$ where $\lambda \in K$. By expanding f^{2h} we can see that

$$xf^{2h} = x(x^{6h} + \cdots + y^{6h}) = (x^{2q-1} + \cdots + xy^{2q-2}),$$

which is certainly not contained in $(x^{2q}, x^q y^q, y^{2q}) = (x^2, xy, y^2)^{[q]}$. Similarly, $xf^{2h} \notin (x^q, y^{2q})$ since $x(\cdots + y^{6h}) \notin (x^{3h+1}, y^{6h+2})$. Also, $xf^{2h} \notin (x^{2q}, y^q)$ since $x(x^{6h} + \cdots) \notin (x^{6h+2}, y^{3h+1})$. Now suppose $xf^{2h} \in (x^{2q}, x^q y^q, y^{2q}, x^q + \lambda^q y^q)$. Make a change of variables and replace x by $x - \lambda y$. Now it is sufficient to show that

$$(x - \lambda y)[(x - \lambda y)^3 + y^3]^{2h} \in ((x - \lambda y)^{2q}, (x - \lambda y)^q y^q, y^{2q}, x^q).$$

This is equivalent to having

$$(x - \lambda y)(x^3 - \lambda x^2 y + \lambda^2 x y^2 - \lambda^3 y^3 + y^3)^{2h} \in (x^{2q} + \lambda^{2q} y^{2q}, x^q y^q - \lambda^q y^{2q}, y^{2q}, x^q).$$

This implies that

$$(x - \lambda y)(x^3 - \lambda x^2 y + \lambda^2 x y^2 - \lambda^3 y^3 + y^3)^{2h} \in (x^q, y^{2q}).$$

But

$$\begin{aligned}(x-\lambda y)(x^3-\lambda x^2y+\lambda^2xy^2-\lambda^3y^3+y^3)^{2h} &= (x-\lambda y)(\,\cdots+\lambda^{4h}x^{2h}y^{4h}+\cdots\,)\\ &= (\,\cdots+\lambda^{4h}x^{2h+1}y^{4h}+\cdots\,),\end{aligned}$$

which is clearly not contained in $(x^q, y^{2q}) = (x^{3h+1}, y^{6h+2})$. Thus $uz^2 \notin I^*$.

Proof of (3). In this case $I = \big(H, H\colon(x,y), H\colon f\big)$. Assume $q = 3h+1$. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.1) we know that

$$I\colon(x,y,z) = \big(H\colon(x,y)\big) + \big(H\colon(x^2,xy,y^2)\big)z + (H\colon f)z^2.$$

So the socle has components in R_0 and R_1. Let $u \in \big(H\colon(x,y)\big)\backslash H$. Then u represents an element of the socle mod I and u is in R_0. If $u \in I^*$, then, using x as a test element, we must have

$$xu^q \in H^{[q]} + \big(H\colon(x,y)\big)^{[q]}f^{h+1} + (H\colon f)^{[q]}f^{2h+1}$$

in $K[[x,y]]$ (Lemma 4.4.3). Multiplying by f^h gives

$$xu^qf^h \in H^{[q]}f^h + \big(H\colon(x,y)\big)^{[q]}f^{2h+1} + (H\colon f)^{[q]}f^q.$$

Using the fact that we are working over a regular ring, we can write this as

$$xu^qf^h \in H^{[q]}f^h + \big(H^{[q]}\colon(x,y)^{[q]}\big)f^{2h+1} + (H^{[q]}\colon f^q)f^q$$

(Corollary 4.4.8), or

$$xu^qf^h \in H^{[q]}f^h + \big(H^{[q]}\colon(x^q,y^q)\big)f^{2h+1} + H^{[q]}.$$

Since $f^{2h+1} \in (x^q, y^q)$ (Lemma 4.4.4), we know that

$$xu^q f^h \in H^{[q]} f^h + H^{[q]} + H^{[q]},$$

which certainly implies that $xu^q f^h \in H^{[q]}$. This, in turn, implies that $xf^h \in H^{[q]}\colon u^q$. Again, using the fact that we are working over a regular ring, we see that $xf^h \in (H\colon u)^{[q]}$ (Corollary 4.4.8). Now $(x, y) \subseteq H\colon u$, and since $u \notin H$, $H\colon u$ must be a proper ideal. In other words, $H\colon u = (x, y)$. So now we know that $xf^h \in (x, y)^{[q]} = (x^q, y^q)$. But $xf^h \notin (x^q, y^q)$. To see this just expand $f^h = x^{3h} + \cdots + y^{3h}$ and see that $xf^h = (\cdots + xy^{3h}) \notin (x^{3h+1}, y^{3h+1}) = (x^q, y^q)$. Thus $u \notin I^*$.

Let $u \in \big(H\colon (x^2, xy, y^2)\big) \backslash (H\colon (x, y))$. Then uz represents an element of the socle mod I and uz is in R_1. If $uz \in I^*$, then, using x as a test element, we must have

$$xu^q f^h \in H^{[q]} + \big(H\colon (x, y)\big)^{[q]} f^h + (H\colon f)^{[q]} f^{2h+1}$$

in $K[[x, y]]$ (Lemma 4.4.3). Multiplying by f^h gives

$$xu^q f^{2h} \in H^{[q]} f^h + \big(H\colon (x, y)\big)^{[q]} f^{2h} + (H\colon f)^{[q]} f^q.$$

Using the fact that we are working over a regular ring, we can write this as

$$xu^q f^{2h} \in H^{[q]} f^h + \big(H^{[q]}\colon (x, y)^{[q]}\big) f^{2h} + (H^{[q]}\colon f^q) f^q$$

(Corollary 4.4.8), or

$$xu^q f^{2h} \in H^{[q]} f^h + \big(H^{[q]}\colon (x^q, y^q)\big) f^{2h} + H^{[q]}.$$

Since $f^{2h} \in (x^q, y^q)$ (Lemma 4.4.4), we know that

$$xu^q f^{2h} \in H^{[q]} f^h + H^{[q]} + H^{[q]},$$

which certainly implies that $xu^q f^{2h} \in H^{[q]}$. This, in turn, implies that $xf^{2h} \in H^{[q]} \colon u^q$. Again, using the fact that we are working over a regular ring, we see that $xf^{2h} \in (H \colon u)^{[q]}$ (Corollary 4.4.8). But this cannot happen by the second part of case (2). Thus $uz \notin I^*$.

Proof of (4). In this case $I = \big(H, H \colon (x,y), H \colon (x,y)\big)$. Let $q = 3h+1$. Using the $\mathbb{Z}_3$-grading we know that

$$I \colon (x,y,z) = \big(H \colon (x,y)\big) + \big(H \colon (x,y)\big)z + \big(H \colon (x^2,xy,y^2)\big)z^2$$

(Lemma 4.4.1). So the socle has components in R_0 and R_2. Let $u \in \big(H \colon (x,y)\big) \backslash H$. Then u represents an element of the socle mod I and u is in R_0. If $u \in I^*$, then, using x as a test element, we must have

$$xu^q \in H^{[q]} + \big(H \colon (x,y)\big)^{[q]} f^{h+1} + \big(H \colon (x,y)\big)^{[q]} f^{2h+1}$$

in $K[[x,y]]$ (Lemma 4.4.3). Multiplying by f^h gives

$$xu^q f^h \in H^{[q]} f^h + \big(H \colon (x,y)\big)^{[q]} f^{2h+1} + \big(H \colon (x,y)\big)^{[q]} f^q.$$

Using the fact that we are working over a regular ring, we can write this as

$$xu^q f^h \in H^{[q]} f^h + \big(H^{[q]} \colon (x,y)^{[q]}\big) f^{2h+1} + \big(H^{[q]} \colon (x,y)^{[q]}\big) f^q$$

(Corollary 4.4.8), or

$$xu^q f^h \in H^{[q]} f^h + \big(H^{[q]} \colon (x^q,y^q)\big) f^{2h+1} + \big(H^{[q]} \colon (x^q,y^q)\big) f^q.$$

Since $f^{2h+1} \in (x^q,y^q)$ (Lemma 4.4.4), we know that

$$xu^q f^h \in H^{[q]} f^h + H^{[q]} + H^{[q]},$$

which certainly implies that $xu^q f^h \in H^{[q]}$. This, in turn, implies that $xf^h \in H^{[q]}\colon u^q$. Again, using the fact that we are working over a regular ring, we see that $xf^h \in (H\colon u)^{[q]}$. Now $(x, y) \subseteq H\colon u$, and since $u \notin H$, $H\colon u$ must be a proper ideal. In other words, $H\colon u = (x, y)$. So now we know that $xf^h \in (x, y)^{[q]} = (x^q, y^q)$. But $xf^h \notin (x^q, y^q)$. To see this just expand $f^h = x^{3h} + \cdots + y^{3h}$ and see that $xf^h = (\cdots + xy^{3h}) \notin (x^{3h+1}, y^{3h+1})$. Thus $u \notin I^*$.

Let $u \in \big(H\colon (x^2, xy, y^2)\big)\backslash\big(H\colon (x, y)\big)$. Then uz^2 represents an element of the socle mod I and uz^2 is in R_2. If $uz^2 \in I^*$, then, using x as a test element, we must have

$$xu^q f^{2h} \in H^{[q]} + \big(H\colon (x, y)\big)^{[q]} f^h + \big(H\colon (x, y)\big)^{[q]} f^{2h}$$

in $K[[x, y]]$ (Lemma 4.4.3). Multiplying by f^{h-2} gives

$$xu^q f^{3h-2} \in H^{[q]} f^{h-2} + \big(H\colon (x, y)\big)^{[q]} f^{2h-2} + \big(H\colon (x, y)\big)^{[q]} f^{3h-2}.$$

Using the fact that we are working over a regular ring, we can write this as

$$xu^q f^{3h-2} \in H^{[q]} f^{h-2} + \big(H^{[q]}\colon (x, y)^{[q]}\big) f^{2h-2} + \big(H^{[q]}\colon (x, y)^{[q]}\big) f^{3h-2}$$

(Corollary 4.4.8), or

$$xu^q f^{3h-2} \in H^{[q]} f^{h-2} + \big(H^{[q]}\colon (x^q, y^q)\big) f^{2h-2} + \big(H^{[q]}\colon (x^q, y^q)\big) f^{3h-2}.$$

Since $f^{2h-2} \in (x^q, y^q)$ (Lemma 4.4.6), we know that

$$xu^q f^{3h-2} \in H^{[q]} f^{h-2} + H^{[q]} + H^{[q]},$$

which certainly implies that $xu^q f^{3h-2} \in H^{[q]}$. This, in turn, implies that $xf^{3h-2} \in H^{[q]}\colon u^q$. Again, using the fact that we are working over a regular ring, we see that $xf^{3h-2} \in (H\colon u)^{[q]}$ (Corollary 4.4.8). Now $(x^2, xy, y^2) \subseteq H\colon u$, and since $u \notin H\colon (x,y)$, $H\colon u \neq (x,y)$. In other words, $(x^2, xy, y^2) \subseteq H\colon u \subsetneq (x,y)$. As in the proof of (2), we know that $H\colon u = (x^2, xy, y^2)$ or (x, y^2) or (x^2, y) or $(x^2, xy, y^2, x + \lambda y)$ where $\lambda \in K$. By expanding f^{3h-2} we can see that

$$xf^{3h-2} = x(\cdots + \tbinom{3h-2}{h-1} x^{3(h-1)} y^{3(2h-1)} + \cdots) = (\cdots + \tbinom{3h-2}{h-1} x^{q-3} y^{2q-5} + \cdots).$$

We know that $\binom{3h-2}{h-1} \not\equiv 0 \mod p$ by Proposition 4.4.6, so

$$(\cdots + \tbinom{3h-2}{h-1} x^{q-3} y^{2q-5} + \cdots) \notin (x^{2q}, x^q y^q, y^{2q}) = (x^2, xy, y^2)^{[q]}.$$

Similarly, $xf^{3h-2} \notin (x^q, y^{2q})$ since

$$x(\cdots + \tbinom{3h-2}{h-1} x^{3(h-1)} y^{3(2h-1)} + \cdots) \notin (x^{3h+1}, y^{6h+2}).$$

Also, $xf^{3h-2} \notin (x^{2q}, y^q)$ since

$$x(\cdots + \tbinom{3h-2}{2h-1} x^{3(2h-1)} y^{3(h-1)} + \cdots) \notin (x^{6h+2}, y^{3h+1}).$$

Now suppose $xf^{3h-2} \in (x^{2q}, x^q y^q, y^{2q}, x^q + \lambda^q y^q)$. Make a change of variables and replace x by $x - \lambda y$. Now it is sufficient to show that

$$(x - \lambda y)[(x - \lambda y)^3 + y^3]^{3h-2} \in ((x - \lambda y)^{2q}, (x - \lambda y)^q y^q, y^{2q}, x^q).$$

This is equivalent to having

$$(x - \lambda y)(x^3 - \lambda x^2 y + \lambda^2 x y^2 - \lambda^3 y^3 + y^3)^{3h-2} \in (x^{2q} + \lambda^{2q} y^{2q}, x^q y^q - \lambda^q y^{2q}, y^{2q}, x^q),$$

which implies that

$$(x - \lambda y)(x^3 - \lambda x^2 y + \lambda^2 x y^2 - \lambda^3 y^3 + y^3)^{3h-2} \in (x^q, y^{2q}).$$

But

$$\begin{aligned}(x-\lambda y)(x^3-\lambda x^2y+\lambda^2xy^2-\lambda^3y^3+y^3)^{3h-2} &= (x-\lambda y)(\cdots+\lambda^{6h-4}x^{3h-2}y^{6h-4})\\ &= (\cdots+\lambda^{6h-4}x^{3h-1}y^{6h-4}),\end{aligned}$$

which is clearly not contained in $(x^{3h+1}, y^{6h+2}) = (x^q, y^{2q})$. Thus $uz^2 \notin I^*$.

Proof of (5). In this case $I = \big(H, H, H\colon (x^2, y)\big)$. Let $p = 3h+2$. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.1) we know that

$$I\colon (x,y,z) = H + \big(H\colon (x,y)\big)z + \big(H\colon (x^3, xy, y^2)\big)z^2.$$

So the socle has components in R_1 and R_2. Let $u \in \big(H\colon (x,y)\big)\backslash H$. Then uz represents an element of the socle mod I and uz is in R_1. If $uz \in I^*$, then, using x as a test element, we must have

$$xu^pf^h \in H^{[p]} + H^{[p]}f^h + \big(H\colon (x^2,y)\big)^{[p]}f^{2h+1}$$

in $K[[x,y]]$ (Lemma 4.4.2). Let $A = K[[x,y]]$. Taking pth roots of both sides yields

$$(*) \qquad x^{1/p}uf^{h/p} \in HA^{1/p} + Hf^{h/p}A^{1/p} + \big(H\colon (x^2,y)\big)f^{(2h+1)/p}A^{1/p}.$$

Next we want to see that $x^{1/p}f^{h/p}$ is part of a free basis for $A^{1/p}$ over A. This is equivalent to showing that xf^h is part of a free basis for A over $A^p = K[[x^p, y^p]]$. For this it is sufficient to see that xf^h is not in the expansion of the maximal ideal of A to A^p. If we expand $f^h = x^{3h} + \cdots + y^{3h}$, it is clear that $x(x^{3h} + \cdots + y^{3h}) \notin (x^p, y^p)$ as $xy^{3h} \notin (x^p, y^p)$. Since $x^{1/p}f^{h/p}$ is part of a free basis for $A^{1/p}$ over A, we have an A-linear map $\theta\colon A^{1/p} \to A$, sending $x^{1/p}f^{h/p}$ to 1. It is clear that

$\theta(f^{h/p}A^{1/p}) \subseteq A$. If we expand $f^{(2h+1)/p}$ and write it in terms of the basis, we see that $\theta(f^{(2h+1)/p}A^{1/p}) \subseteq (x^2, xy, y^2)A$. Thus applying θ to $(*)$ gives

$$u \in H + H + \big(H\colon (x^2, y)\big)(x^2, xy, y^2).$$

Since $(x^2, xy, y^2) \subseteq (x^2, y)$, this implies that $u \in H$ which is a contradiction. Hence $uz \notin I^*$.

Now let $u \in \big(H\colon (x^3, xy, y^2)\big) \backslash \big(H\colon (x^2, y)\big)$. So uz^2 represents an element of the socle mod I and uz^2 is in R_2. If $uz^2 \in I^*$, then, using y as a test element, we must have

$$yu^p f^{2h} \in H^{[p]} + H^{[p]}f^h + \big(H\colon (x^2, y)\big)^{[p]} f^{2h}$$

in $K[[x, y]]$ (Lemma 4.4.2). Taking pth roots of both sides yields

$$(**) \qquad y^{1/p}uf^{2h/p} \in HA^{1/p} + Hf^{h/p}A^{1/p} + \big(H\colon (x^2, y)\big)f^{2h/p}A^{1/p}.$$

Next we want to see that $y^{1/p}f^{2h/p}$ is part of a free basis for $A^{1/p}$ over A. As before, it is sufficient to see that yf^{2h} is not in the expansion of the maximal ideal of A to A^p. First, expand $f^{2h} = x^{6h} + \cdots + y^{6h}$. The middle term of the expansion is $\binom{2h}{h}x^{3h}y^{3h}$. If $p = 3h+2$, then $\binom{2h}{h} \not\equiv 0 \mod p$ (see the proof of Lemma 4.4.4). So $y(x^{6h} + \cdots + y^{6h}) \notin (x^p, y^p)$ as $\binom{2h}{h}x^{3h}y^{3h+1} \notin (x^p, y^p)$. Since $y^{1/p}f^{2h/p}$ is part of a free basis for $A^{1/p}$ over A, we have an A-linear map $\theta\colon A^{1/p} \to A$, sending $y^{1/p}f^{2h/p}$ to 1. It is clear that $\theta(f^{h/p}A^{1/p}) \subseteq A$. If we expand $f^{2h/p}$ and write it in terms of the basis, we see that $\theta(f^{2h/p}A^{1/p}) \subseteq (x^2, y)A$. Thus applying θ to $(**)$ gives

$$yu \in H + H + \big(H\colon (x^2, y)\big)(x^2, y).$$

This implies that $yu \in H$ or $u \in (H\colon y) \subseteq H\colon (x^2, y)$ which is a contradiction. Hence $uz^2 \notin I^*$. $\square$

In addition, in the following cases we can prove that if $u \in I^*$, then $u \in I^F$ for some but not all elements of the socle.

(4.5.2) Proposition. *Let* I *be a* $\mathbb{Z}_3$*-graded ideal of* $K[[x, y, z]]/(x^3+y^3+z^3)$*, where* K *is a field of characteristic* p *and* $p \equiv 2 \mod 3$*.*

(1) *If* $I = \big(H, H, H\colon (x^2, xy, y^2)\big)$*, then* $uz \notin I^*$ *where* $u \in H\colon (x, y)$*.*

(2) *If* $I = (H, J, J)$*, then* $u \in I^*$ *implies* $u \in I^F$ *where* $u \in (H\colon (x, y))\backslash H$*.*

(3) *If* $I = (H, H, J)$*, then* $uz^2 \in I^*$ *implies* $uz^2 \in I^F$ *where* $u \in H\colon (x, y)$*.*

Proof. Proof of (1). Let $p = 3h + 2$. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.1) we know that

$$I\colon (x, y, z) = H + \big(H\colon (x, y)\big)z + \big(H\colon (x, y)^3\big)z^2.$$

So the socle has components in R_1 and R_2. Let $u \in \big(H\colon (x, y)\big)\backslash H$. Then uz represents an element of the socle mod I and uz is in R_1. If $uz \in I^*$, then, using x as a test element, we must have

$$xu^p f^h \in H^{[p]} + H^{[p]} f^h + \big(H\colon (x, y)^2\big)^{[p]} f^{2h+1}$$

in $K[[x, y]]$ (Lemma 4.4.2). Taking pth roots yields

$$(*) \qquad x^{1/p} u f^{h/p} \in HR^{1/p} + Hf^h R^{1/p} + \big(H\colon (x, y)^2\big) f^{(2h+1)/p} R^{1/p}.$$

Let $A = K[[x, y]]$. We know from Proposition 4.5.1 (5) that we have an A-linear map $\theta\colon A^{1/p} \to A$, sending $x^{1/p} f^{h/p}$ to 1. It is clear that $\theta(f^{h/p} A^{1/p}) \subseteq A$. If we expand $f^{2h+1/p}$ and write it in terms of the basis, we see that $\theta(f^{2h+1/p}A^{1/p}) \subseteq (x^2, xy, y^2)A$. Thus applying θ to $(*)$ gives

$$u \in H + H + \big(H\colon (x,y)^2\big)(x,y)^2,$$

or $u \in H$, which is a contradiction. Hence $uz \notin I^*$.

A similar technique does not work when trying to determine whether a socle element in R_2 is in I^*.

Proof of (2). Let $q = 3h+2$. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.1) we know that

$$I\colon (x,y,z) = \big((H\colon (x,y)) \cap J\big) + Jz + \big((J\colon (x,y)) \cap (H\colon (x^3+y^3))\big)z^2.$$

Let $u \in \big((H\colon (x,y)) \cap J\big)\backslash H$, so u represents an element of the socle mod I and $u \in R_0$. If $u \in I^*$, then, using z as a test element, $zu^q \in I^{[q]}$. Using the grading (Lemma 4.4.2), we determine that this is equivalent to having

$$u^q \in H^{[q]} + J^{[q]}f^{h+1} + J^{[q]}f^{2h+1},$$

which certainly implies that $u^q \in H^{[q]} + J^{[q]}f^{h+1}$. In order to have $u \in I^F$, we need $u^q \in I^{[q]}$ for $q \gg 0$, or equivalently,

$$u^q \in H^{[q]} + J^{[q]}f^{h+1} + J^{[q]}f^{2h+2}$$

(Lemma 4.4.2). But $u^q \in H^{[q]} + J^{[q]}f^{h+1}$ implies that

$$u^q \in H^{[q]} + J^{[q]}f^{h+1} + J^{[q]}f^{2h+2},$$

so if $u \in I^*$, then $u \in I^F$. The other contribution to the socle mod I is uz^2 where $u \in \big((J\colon (x,y)) \cap (H\colon f)\big)\backslash J$. Unfortunately, this technique provides no information in this case.

Proof of (3). Let $q = 3h + 2$. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.1) we know that

$$I\colon (x, y, z) = H + \big((H\colon (x, y)) \cap J\big)z + \big((J\colon (x, y)) \cap (H\colon (x^3 + y^3))\big)z^2.$$

Let $u \in \big((J\colon (x, y)) \cap (H\colon (x^3 + y^3))\big) \backslash H$, so uz^2 represents the socle mod I and uz^2 is in R_2. If $uz^2 \in I^*$, then, using z as a test element, we must have $z(uz^2) \in I^{[q]}$. This implies that

$$zu^q z^{2q} \in I^{[q]},$$

or

$$u^q f^{2h+1} z^2 \in I^{[q]}.$$

Using the grading we see that this is equivalent to showing that

$$u^q f^{2h+1} \in H^{[q]} + H^{[q]} f^h + J^{[q]} f^{2h+1}$$

in $K[[x, y]]$ (Lemma 4.4.2). This certainly implies that

$$u^q f^{2h+1} \in H^{[q]} + J^{[q]} f^{2h+1}.$$

In order to have $uz^2 \in I^F$, we need $(uz^2)^q \in I^{[q]}$. Using the grading (Lemma 4.4.2) we see that this is equivalent to having

$$u^q f^{2h+1} \in H^{[q]} + H^{[q]} f^{h+1} + J^{[q]} f^{2h+1}.$$

If $u^q f^{2h+1} \in H^{[q]} + J^{[q]} f^{2h+1}$, then $u^q f^{2h+1} \in H^{[q]} + H^{[q]} f^{h+1} + J^{[q]} f^{2h+1}$. In other words, if $uz^2 \in I^*$, then $uz^2 \in I^F$. As before, this technique provides no information about the contribution to the socle from R_1. $\square$

CHAPTER 5

LARGE INJECTIVES

We can study the question of whether $I^* = I^F$ in a ring R from another point of view. The problem can be approached by looking at injective modules over R^∞. For example, if it were true that one could write the injective hull of K over R^∞ as a direct limit of cyclic modules, R^∞/I_ν, then we could reduce the problem for modules to studying the ideals I_ν. Ultimately, we would like to determine the injective hull of the residue field over R^∞ for $R = K[[x,y,z]]/(x^3+y^3+z^3)$. At this point we can find a $\mathbb{Z}_3$-graded injective R^∞-module that contains a copy of K. This is enough to give certain reductions in the problem of whether tight closure is the same as plus closure.

We begin by determining the injective hull for $K[x]^\infty$.

(5.1.1) Proposition. *Let K be a field of characteristic p and let $E = \{\sum \lambda_\alpha x^\alpha\colon \alpha \in \mathbb{Z}[1/p], \alpha \le 0, \lambda_\alpha \in K$, the set of all α occuring is well-ordered under $\le\}$. Then E is the injective hull of K over $K[x]^\infty$.*

Proof. First note that $K[x]^\infty_m = \cup_q K[x^{1/q}]_m$ and $K[x^{1/q}]_m$ is a valuation ring over $\mathbb{Z}[1/p]$ for all q. $K[x^{1/q}]_m$ has two kinds of ideals. One class of ideals is of the form $(x^\beta) = (x^\alpha\colon \alpha \ge \beta, \beta \in \mathbb{Z}[1/p])$. The other type of ideal is of the form $I_\beta = (x^\alpha\colon \alpha > \beta, \beta \in \mathbb{R} \ge 0)$.

By requiring that exponents of monomials in E be negative, we can keep track of the module structure. We multiply in the obvious way and kill any terms that arise involving positive powers of x.

Suppose we have a map $\phi : I \to E$ for some ideal I. We need to show that this extends to a map $K[x]^\infty \to E$. To do this we need to specify where 1 is mapped. Suppose $I = (x^\beta)$ and $x^\beta \mapsto v$. In order to extend the map, we need to find u such that $x^\beta u = v$ and then we let u be the image of 1. If $v = \sum \lambda_\alpha x^\alpha$, we may take $u = \sum \lambda_\alpha x^{\alpha-\beta}$.

Suppose $I = I_\beta$ and $x^{\beta_0} \mapsto e_0$, $x^{\beta_1} \mapsto e_1$, etc., where $\{\beta_0, \beta_1, \dots\}$ is sequence in $\mathbb{R} \geq 0$ approaching β. Again, in order to extend the map, we must determine the image of 1. Since $x^{\beta_i - \beta_{i+1}} \cdot x^{\beta_{i+1}} = x^{\beta_i}$, we must have $x^{\beta_i - \beta_{i+1}} \cdot e_{i+1} = e_i$. Multiplying by $x^{\beta_i - \beta_{i+1}}$ shifts the exponents of e_{i+1} and kills any term that involves a positive exponent after the multiplication. We can write $e_{i+1} = a_{i+1} + b_{i+1}$ where $a_{i+1} = a_i / x^{\beta_i - \beta_{i+1}}$ and $x^{\beta_i - \beta_{i+1}} \cdot b_{i+1} = 0$. Then we will map 1 to $e_0/x^{\beta_0} + b_1/x^{\beta_1} + b_2/x^{\beta_2} + \cdots$. This construction ensures that $\phi(x^\beta) = x^\beta \phi(1)$. It remains to show that the image of 1 is in E. To do this we must show that the set of exponents occuring is a well-ordered set. The support of the image of 1 is

$$\operatorname{Supp}(e_0/x^{\beta_0}) \cup \operatorname{Supp}(b_1/x^{\beta_1}) \cup \operatorname{Supp}(b_2/x^{\beta_2}) \cup \cdots = A_0 \cup A_1 \cup \cdots$$

where the support of e_0/x^{β_0}, for example, is just $\{\, \alpha - \beta_0 \colon \alpha \in \operatorname{Supp}(e_0) \,\}$. All elements of A_{i+1} exceed all elements of A_i, and each A_i is well-ordered, so the support of the image of 1 is well-ordered. Hence the image of 1 is contained in E.

It is easy to see that $K \to E$ is an essential extension. Suppose $e = \sum \lambda_\alpha x^\alpha \in E$

and α_0 is the least α occuring. Then $x^{-\alpha_0} \cdot e \in K$. $\square$

Next we will consider injectives over $K[x, y]^\infty$. Unfortunately, we cannot yet determine the injective hull in this case.

(5.1.2) Discussion. The obvious analogue of the of construction in Proposition 5.1.1 for $K[x, y]^\infty$ does not yield the injective hull of the residue field. In (5.1.1) we considered formal sums of monomials such that the exponents occuring are well-ordered. For the two-variable case, we will consider formal sums of monomials such that the exponents occuring, considered as a subset of $\mathbb{Q} \times \mathbb{Q}$, are well-founded. We call a partially ordered set *well founded* (or *Artinian*) if every non-empty subset has a minimal element. This means exactly that every strictly decreasing sequence is finite. We take the partial order on $\mathbb{Q}\times\mathbb{Q}$ to be coordinatewise: $(a, b) \le (c, d)$ if $a \le c$ and $b \le d$. We can show that $E = \{\sum \lambda_{\alpha,\beta} x^\alpha y^\beta \colon \alpha, \beta \in \mathbb{Z}[1/p], \alpha, \beta \le 0, \lambda_{\alpha,\beta} \in K,$ the set of all (α, β) occuring is well-founded $\}$ is an essential extension of K over $K[x, y]^\infty$. In fact, E is also a module over $K[[x, y]]^\infty$ (it suffices to check that re has only finitely many terms in any degree where $r \in K[[x, y]]^\infty$ and $e \in E$). However, E admits an essential extension and so it is not injective.

Next we will consider injectives over $K[[x, y, z]]/(x^3 + y^3 + z^3)$. Again, we cannot yet determine the injective hull, but we can find a $\mathbb{Z}_3$-graded injective containing a copy of K. We will use the following general lemma.

(5.1.3) Lemma. *If R is an A-algebra and E is injective over A, then* $\operatorname{Hom}_A(R, E)$ *is an injective R-module.*

Proof. See [E, Lemma A3.8].

(5.1.4) Comment. With $R = K[[x,y,z]]/(x^3+y^3+z^3)$, $A = K$ and $E = K$, we see that $E_{R^\infty} = \operatorname{Hom}_K(R^\infty, K)$ is an injective R^∞-module.

In order to use this injective to reduce the problem of whether $I^* = I^F$ to the graded irreducible case, we will show that it contains a copy of K and that it is $\mathbb{Z}_3$-graded.

(5.1.5) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$ and $E_{R^\infty} = \operatorname{Hom}_K(R^\infty, K)$. Then $K \hookrightarrow E_{R^\infty}$.*

Proof. Let $\phi \in \operatorname{Hom}_K(R^\infty, K)$ be the map $\phi\colon R^\infty \to R^\infty/m_{R^\infty} \hookrightarrow K$. Then $R^\infty\phi \cong K$, since $m_{R^\infty}\phi(x) = \phi(m_{R^\infty}x) = 0$. $\square$

Next we would like to see that E_{R^∞} is $\mathbb{Z}_3$-graded.

(5.1.6) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$ and $E_{R^\infty} = \operatorname{Hom}_K(R^\infty, K)$. Then E_{R^∞} is $\mathbb{Z}_3$-graded.*

Proof. Recall that the grading on R extends to R^∞ (see Section 4.1). Next, to see that E_{R^∞} is graded, write $R^\infty = R_0 + R_1 + R_2$. Then

$$\begin{aligned}\operatorname{Hom}_K(R^\infty,K) &= \\ &\operatorname{Hom}_K(R_0, K) \oplus \operatorname{Hom}_K(R_1, K) \oplus \operatorname{Hom}_K(R_2, K).\end{aligned}$$

Let $E_{R^\infty} = W_0 + W_1 + W_2$ where $W_i = \operatorname{Hom}_K(R_{2i}, K)$. Any subscripts that indicate a graded piece of a module or ring, e.g. $2i$, will be reduced mod 3. If $\phi_i \in W_i$ and $r_i \in R_i$, then $\phi_i(r_i) \in K$ and $\phi_i(r_j) = 0$ when $i \neq j$.

Let $f_i \in R_i$ and $\phi_j \in W_j$. We want to see that $f_i\phi_j \in W_{i+j}$. Recall that $W_{i+j} = \operatorname{Hom}_K(R_{2(i+j)}, K)$, so we need to show that $f_i\phi_j \in \operatorname{Hom}_K(R_{2(i+j)}, K)$.

Since $f_i\phi_j(r_{2(i+j)}) = \phi_j(f_ir_{2(i+j)})$ and $f_ir_{2(i+j)} \in R_{i+2(i+j)} = R_{3i+j} = R_j$, we know that $f_i\phi_j(r_{2(i+j)}) \in W_{i+j}$ as required. Similarly, if $k \neq 2(i+j)$, then $f_i\phi_j(r_k) = 0$ and hence $f_i\phi_j \in W_{i+j}$. $\square$

(5.1.7) Theorem (Reduction to $\mathbb{Z}_3$-graded module case). *Let* $R = K[[x,y,z]]/(x^3+y^3+z^3)$, *where* K *is a field of characteristic* p. *Let* $I \subseteq R$ *be an* m*-primary ideal such that* $I^* \neq I^F$. *Then there exists a* $\mathbb{Z}_3$*-graded* R*-module* M *and an irreducible* $\mathbb{Z}_3$*-graded submodule* N *such that* $N^* \neq N^F$.

Proof. Suppose $I \subseteq R$ is an m-primary ideal such that $I^* \neq I^F$. Then there exists $u \in I^*R^\infty \backslash IR^\infty$. Expand IR^∞ to an ideal of R^∞ maximal with respect to not containing u. Then u is the socle mod IR^∞ and IR^∞ is irreducible. To see that $um_{R^\infty} = 0$, note that $m_{R^\infty} = \cup m_{R^{1/q}}$. Also, $u \in (I \cap R^{1/q})^*$ for some q. This implies that $m_{R^{1/q}}u \subseteq I \cap R^{1/q}$. Thus $m_{R^\infty}u \subseteq IR^\infty$.

Let E_{R^∞} be a $\mathbb{Z}_3$-graded injective R^∞-module that contains a copy of K. We know one exists by Lemmas 5.1.5 and 5.1.6. We have the following inclusion

$$\begin{array}{ccc} R^\infty/IR^\infty & \longrightarrow & E_{R^\infty} \\ 1 & \longmapsto & \alpha \end{array}$$

We can find a finitely generated ideal $I_0 \subseteq R^{1/q}$ such that $u \in I_0^{*fg}$. Let $\tilde{u}$ be the image of u in $R^{1/q}/I_0$. Let M be the submodule of E_{R^∞} generated by α. Then we have a map

$$R^{1/q}/I_0 \longrightarrow M.$$

M is a finitely generated $R^{1/q}$-module that contains the image of $R^{1/q}/I_0$ and is graded. It is still true that $\tilde{u} \in I_0^*$ in M since $u \in 0^*$ in E_{R^∞}. If $u \in 0^F$ in M, then we would have $u \in 0^F$ in E_{R^∞} and an element is in 0^F in an R^∞-module if and only if it is zero. Thus $u \notin 0^F$ in M. $\square$

The existence of injective R^∞-modules with other gradings would give similar reductions. For example, it would be very helpful to find an $\mathbb{N}$-graded injective as a reduction to the $\mathbb{N}$-graded case would simplify the problem greatly.

CHAPTER 6

IRREDUCIBLE GRADED IDEALS

As we saw in Chapter 5, we can reduce the question of whether $I^* = I^F$ in $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$ to the graded irreducible module case. Given this reduction, it seems likely that understanding the graded irreducible ideal case will be helpful. In this chapter we will show that $I^* = I^F$ for most $\mathbb{Z}_3$-graded irreducible ideals in R when K has characteristic p and $p \equiv 2 \mod 3$. We will make use of the $\mathbb{Z}_3$-grading and many of the techniques introduced in Chapter 4.

In the course of proving the main result, Theorem 6.3.1, we develop a number of techniques for determining when an element of the socle is in the tight closure or the Frobenius closure of a given ideal. Given the difficulty of calculating tight closure in general, these techniques provide potentially useful concrete information.

6.1 Preliminary Techniques

The following proposition provides a useful tool for determining whether or not a given irreducible m-primary ideal, I, is tightly closed. If we can find an irreducible ideal contained in I which is tightly closed, then we know that I is also tightly closed. Similarly, if we can find an irreducible ideal containing I which is not tightly closed, then we know that I is not tightly closed.

(6.1.1) Proposition. *Let R be a local Gorenstein ring. Let m be the maximal ideal of R and let J and I be irreducible m-primary ideals of R with $J \subseteq I$. Then $R/I \hookrightarrow R/J$, and if $I^* \neq I$, then $J^* \neq J$. Also, if $I^F \neq I$, then $J^F \neq J$.*

Proof. Since I and J are m-primary, R/I and R/J are zero-dimensional. As I and J are irreducible and m-primary, $\dim_K \operatorname{Soc} R/J = 1$ and R/J is Gorenstein, and similarly for R/I. So R/J is a zero-dimensional Gorenstein local ring, which implies that R/J is injective as a module over itself and $R/J \cong E_{R/J}(K)$. Similarly, $R/I \cong E_{R/I}(K)$. So

$$\operatorname{Ann}_{R/J} I \cong \operatorname{Ann}_{E_{R/J}(K)} I \cong E_{(R/J)/I}(K) \cong E_{R/I}(K) \cong R/I,$$

and thus $\operatorname{Ann}_{(R/J)} I \cong R/I$. Composing this isomorphism with the natural inclusion $\operatorname{Ann}_{(R/J)} I \hookrightarrow R/J$ gives the inclusion $\phi\colon R/I \to R/J$. We also know that $\phi((0)^*_{R/I}) \subseteq (0)^*_{R/J}$. If $I^* \neq I$, then $(0)^*_{R/I} = I^*/I \neq 0$, and so $(0)^*_{R/J} \neq 0$. Then $J^*/J \neq 0$ and $J^* \neq J$ as required. The same argument applies for I^F and J^F since $\phi((0)^F_{R/I}) \subseteq (0)^F_{R/J}$. $\square$

In fact, even if one or both of I and J is not irreducible, if we can show that we have an injection $R/I \hookrightarrow R/J$, then $J^* = J$ implies that $I^* = I$. The following lemma gives a criterion for when such an injections exists.

(6.1.2) Lemma. *Let R be a Noetherian ring. Let I and J be ideals of R with $J \subseteq I$, I irreducible and let u be the socle mod I. Then $R/I \hookrightarrow (R/J)^h$ if and only if there exists $v \in R$ such that $vI \subseteq J$ and $vu \notin J$. If, in addition, $J = J^*$, then $I = I^*$.*

Proof. Let $u_1, \dots, u_h$ generate $J\colon I$. Let $\bar{u}_1, \dots, \bar{u}_h$ be the images of the generators in R/J. Then $\bar{u}_1, \dots, \bar{u}_h$ generate $(J\colon I)/J \cong \operatorname{Ann}_{R/J} I$. We have a map $R \to (R/J)^h$ taking $\bar{r}$ to $(r\bar{u}_1, \dots, r\bar{u}_h)$. Now $\bar{r}$ gets mapped to 0 if and only if $r(J\colon I) \subseteq J$. This is equivalent to having $r \in J\colon (J\colon I)$. So the map is injective if and only if $I = J\colon (J\colon I)$. This is equivalent to having $u \notin J\colon (J\colon I)$ or $u(J\colon I) \not\subseteq J$. Finally, this is true if and only if there exists $v \in J\colon I$ such that $uv \notin J$.

Suppose $u \in 0^*_{R/I}$. Then the image of u is contained in $0^*_{R/J}$. Thus if J is tightly closed, so is I. $\square$

(6.1.3) Proposition. *Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let I be an irreducible m-primary ideal of R and let u represent the socle mod I. If $I \subseteq (x, y)$, then $u \in I^F$.*

Proof. Since I and (x, y) are both irreducible m-primary ideals, we have an injection $R/(x, y) \hookrightarrow R/I$ sending z^2, the socle in $R/(x, y)$, to u (Proposition 6.1.1). It is enough to see that $z^2 \in (x, y)^F$, for then $u \in I^F$. For this it is sufficient to show that z^{2p} is contained in (x^p, y^p). Let $p = 3h + 2$. Using the basic relation in R we obtain $z^{6h+4} = (x^3 + y^3)^{2h+1}z$. Using the $\mathbb{Z}_3$-grading we conclude that it is sufficient to show that $(x^3 + y^3)^{2h+1} \in (x^p, y^p)$ (Lemma 4.4.2). Expand using the binomial theorem to see

$$(x^3 + y^3)^{2h+1} = \sum_{i=0}^{2h+1} \tbinom{2h+1}{i} x^{3i} y^{3(2h+1-i)}.$$

The middle terms of the expansion are $x^{3h}y^{3h+3}$ and $x^{3h+3}y^{3h}$ or $x^{p-2}y^{p+1}$ and $x^{p+1}y^{p-2}$. These are certainly in the ideal (x^p, y^p), regardless of the coefficients. Thus $z^2 \in (x, y)^F$. $\square$

We have a similar result for an ideal generated by arbitrary parameters.

(6.1.4) Proposition. *Let* $R = K[[x,y,z]]/(x^3+y^3+z^3)$, *where* K *is a field of characteristic* p *and* $p \equiv 2 \mod 3$. *Let* I *be an irreducible* m*-primary ideal of* R *and let* v *represent the socle in* R/I. *Let* (f,g) *be generated by a system of parameters. If* $I \subseteq (f,g)$, *then* $v \in I^F$.

Proof. Let u represent the socle in $R/(f,g)$. Since I and (f,g) are both irreducible m-primary ideals, we have an injection $R/(f,g) \hookrightarrow R/I$ sending u to v (Proposition 6.1.1). It is enough to see that $u \in (x,y)^F$, for then $v \in I^F$. We know that $(f^q, g^q) \subseteq (x,y)$ for some q. The socle mod (f^q,g^q) is $f^{q-1}g^{q-1}u$. Since (f^q,g^q) is an m-primary irreducible ideal contained in (x,y), we know that $f^{q-1}g^{q-1}u \in (f^q,g^q)^F$ (Proposition 6.1.3). This implies that

$$f^{(q-1)Q}g^{(q-1)Q}u^Q \in (f^{qQ}, g^{qQ})$$

for some $Q = p^e$. Dividing by powers of f and g yields $u^Q \in (f^Q, g^Q)$, and hence $u \in (f,g)^F$. □

6.2 Classification of Irreducibles

The $\mathbb{Z}_3$-grading on R makes it fairly easy to characterize the irreducible ideals. The irreducible $\mathbb{Z}_3$-graded ideals of $K[[x,y,z]]/(x^3+y^3+z^3)$ can be classified as follows:

(6.2.1) Proposition. *Let* I *be an irreducible* m*-primary* $\mathbb{Z}_3$*-graded ideal of* $K[[x,y,z]]/(x^3+y^3+z^3)$, *where* K *is a field of characteristic* p. *Then* I *corresponds to one of the following triples of ideals in* $K[[x,y]]$ *where* H *is an irreducible*

m-primary ideal of $K[[x,y]]$ and $f = x^3 + y^3$:

(1) (H, H, H)

(2) $(H, H{:}\, f, H{:}\, f)$

(3) $(H, H, H{:}\, f)$

Proof. We know that $(H_0 + H_1 z + H_2 z^2){:}\,(x, y, z)$ can be decomposed into graded pieces as follows:

$$\big((H_0{:}\,(x,y)) \cap H_2\big) + \big((H_1{:}\,(x,y)) \cap H_2\big)z + \big((H_2{:}\,(x,y)) \cap \big(H_0{:}\,(x^3+y^3)\big)\big)z^2$$

(Lemma 4.4.1). Suppose u, the socle mod I, is contained in R_0, the zero graded piece of R. Then in order for I to have a one-dimensional socle, there must be no contribution from R_1 or R_2. This requires that $\big(H_1{:}\,(x,y)\big) \cap H_2 = H_1$ and $\big(H_2{:}\,(x,y)\big) \cap (H_0{:}\, f) = H_2$. These conditions imply that $H_1 = H_2$ and $H_2 = H_0{:}\, f$, respectively. To see this, just note that if H_1 were strictly contained in H_2, since $H_1{:}\,(x,y)$ is strictly larger than H_1, their intersection would strictly contain H_1. In other words, I corresponds to the triple $(H_0, H_0{:}\, f, H_0{:}\, f)$. The annihilator of (x,y,z) is now $\big(H_0{:}\,(x,y)\big) \cap (H_0{:}\, f)$. Since $(f) \subseteq (x,y)$, we know that $\big(H_0{:}\,(x,y)\big) \subseteq (H_0{:}\, f)$, and so the intersection is just $H_0{:}\,(x,y)$. The socle is then $\big(H_0{:}\,(x,y)\big) \backslash H_0$ or just the socle mod H_0 in $K[[x,y]]$. Thus, if H_0 is an irreducible ideal of $K[[x,y]]$, then I has a one-dimensional socle and is irreducible.

Next we suppose the socle mod I is contained in R_1. Again, this requires that there be no contribution from R_0 or from R_2. The conditions for this are that $\big(H_0{:}\,(x,y)\big) \cap H_1 = H_0$ and $\big(H_2{:}\,(x,y)\big) \cap (H_0{:}\, f) = H_2$. These, in turn, imply that $H_0 = H_1$ and $H_2 = H_0{:}\, f$. Now I corresponds to the triple $(H_0, H_0, H_0{:}\, f)$.

This time the socle is $\big((H_0\colon(x,y))\backslash H_0\big)z$, and as before, if H_0 is irreducible, then so is I. Similarly, if the socle mod I is contained in R_2, we need $\big(H_0\colon(x,y)\big)\cap H_1 = H_0$ and $\big(H_1\colon(x,y)\big)\cap H_2 = H_1$. These conditions imply that $H_0 = H_1$ and $H_1 = H_2$. Now I corresponds to the triple (H_0, H_0, H_0) and the socle is given by $\big((H_0\colon(x,y))\backslash H_0\big)z^2$. □

6.3 Tight Closure and Frobenius Closure of Irreducible Ideals

Now we can prove the main result of this chapter: In most cases, if I is an irreducible $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, then $I^* = I^F$.

(6.3.1) Theorem. *Let I be an irreducible m-primary $\mathbb{Z}_3$-graded ideal of $K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let $f = (x^3+y^3)$. If I has any of the following forms, then $I^* = I^F$.*

1. (H, H, H)
2. $(H, H\colon f, H\colon f)$
3. $(H, H, H\colon f)$ *and* $f \notin H$
4. $(H, H, H\colon f)$ *and* $f \in H$ *and* H *contains an element with a linear form*

Proof of (1)-(3). First observe that $(H, H, H) \subseteq (x, y)$. The ideals $(H, H\colon f, H\colon f)$ and $(H, H, H\colon f)$ are also contained in (x, y) so long as $f \notin H$. If $f \in H$, then $H\colon f = K[[x,y]] = A$. In that case, $(H, H\colon f, H\colon f) = (H, A, A) = H + Az$ and $(H, H, H\colon f) = (H, H, A) = H + Az^2$. When the ideals are contained in (x, y) we know that $I^* = I^F$ by Proposition 6.1.3. In fact, we know that $I^* \neq I$ in those cases. Also, see Proposition 6.3.12 for an alternate proof in the case (H, H, H).

We will now consider the case $I = (H, H\colon f, H\colon f)$ where $f \in H$. As noted before,

$I = H + Az$ in this case. Let $q = 3h+1$. Suppose $u \in I^*$. Then, using z as a test element, we must have $zu^q \in I^{[q]}$. Using the grading (Lemma 4.4.3), we see that this is equivalent to having

$$u^q \in H^{[q]} + (f^{h+1}) + (f^{2h+1})$$

in $K[[x,y]]$ which implies that

$$u^q \in H^{[q]} + (f^{h+1}).$$

This, however, is exactly what is needed to have $u^q \in I^{[q]}$ (Lemma 4.4.3). Thus $u \in I^F$. $\square$

The proof of (4) requires several different techniques. We begin with an analysis of the possible forms for H.

(6.3.2) Lemma. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let I be a $\mathbb{Z}_3$-graded irreducible ideal of the form $(H, H, H\colon f)$ with $f = (x^3+y^3) \in H$. If H contains an element with a linear form, then H has one of the following forms:*

(1) (x, y)

(2) $(x^2, y - cx)$, $c \in K \backslash \{0\}$

(3) $(x^k, y + x)$, $k \geq 3$

(4) $(x^k, y + x - dx^{k-1})$, $k \geq 3$, $d \in K \backslash \{0\}$

Proof. Let $q = 3h+1$. We can assume that $H \not\subseteq (x, y^3)$ in $K[[x,y]]$, otherwise I would be contained in a parameter ideal of R and we would done by Proposition

6.1.4. Suppose an element of H has a term $\alpha y + \cdots$ with $\alpha \neq 0$. Using Weierstrass preparation, we can find a unique monic associate $u = y - g(x)$. Now $K[[x, y]]/u \cong K[[x]]$, a principal ideal domain. $H/(u)$ is an ideal of $K[[x]]$, and since $K[[x]]$ is a PID, $H/(u) = x^k$ for some k. Lifting back to $K[[x, y]]$ we see that $H = (x^k, y - g(x))$. We can also assume that $x^k \notin (y - g(x), z)$, otherwise I would be contained in the ideal $(y - g(x), z)$ which is a parameter ideal. Suppose

$$x^k \notin (y - g(x), z)$$

in R. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.2) we see that this is equivalent to having

$$x^k \notin (y - g(x), x^3 + y^3)$$

in $K[[x, y]]$, which is equivalent to having

$$x^k \notin (x^3 + g(x)^3)$$

in $K[[x, y]]$ modulo $u = y - g(x)$. In order to have $x^k \notin (x^3, g(x)^3)$, we need the order of $x^3 + g(x)^3$ to be greater than k. Assume $ord_x g(x) \geq 2$ or $c \neq -1$ where $g(x) = cx + \cdots$. If $k = 1$, then $H = (x, y - g(x)) = (x, y)$. If $k = 2$, then $H = (x^2, y - g(x)) = (x^2, y - cx)$.

Now suppose that $k > 2$. We still need the order of $x^3 + g(x)^3$ to be greater than k. We can assume that $ord_x g(x) = 1$ and $g(x) = -x + dx^h + \cdots$. Then

$$\begin{aligned} x^3 + g(x)^3 &= x^3 - x^3 + 3x^2 dx^h + \cdots \\ &= 3dx^{2+h} + \cdots \end{aligned}$$

So we need $h+2>k$. If $k<h+1$, then $(x^k, y-g(x)) = (x^k, y+x)$. If $k=h+1$, then $(x^k, y-g(x)) = (x^k, y+x-dx^{k-1})$. In each case $k \geq 3$. $\square$

We can now deal with these cases separately.

(6.3.3) Remark. Let R be a Noetherian ring and m a maximal ideal. If I is an m-primary ideal of R, then $R/I \cong \hat{R}/I\hat{R}$. If we are interested in whether $u \in I\hat{R}$, it is sufficient to check whether $u \in I$. We will make use of this idea in several of the following propositions by reducing questions about ideal membership in $K[[x,y]]$ to the polynomial ring $K[x,y]$.

(6.3.4) Proposition. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let $I = (x,y,z^2)$. Then $I^* = I^F = I$.*

Proof. The socle mod I is z. Using z as a test element, it suffices to see that $zz^p \notin (x^p, y^p, z^{2p})$. Let $p = 3h+2$ and $f = x^3+y^3$. Suppose $zz^p \in (x^p, y^p, z^{2p})$. This is equivalent to having $z^{3h+3} \in (x^p, y^p, z^{6h+4})$ or $(z^3)^{h+1} \in (x^p, y^p, (z^3)^{2h+1}z)$. Using the basic relation in R, we see that this equivalent to having

$$f^{h+1} \in (x^p, y^p, f^{2h+1}z).$$

Using the $\mathbb{Z}_3$-grading (Lemma 4.4.2), we see that this is equivalent to having

$$f^{h+1} \in (x^p, y^p, f^{2h+2})$$

in $K[[x,y]]$. The degree of f^{2h+2} is greater than the degree of f^{h+1}. Since we are in the homogeneous case, we may conclude that

$$f^{h+1} = Ax^p + By^p$$

where $A,\ B \in K[x,y]$ (6.3.3). The degree of f^{h+1} is $p+1$, so we actually know that

$$f^{h+1} = (a_1x + a_2y)x^p + (b_1x + b_2y)y^p$$

where $a_1,\ a_2,\ b_1,\ b_2 \in K$. This equality cannot hold since the term $(h+1)x^{3h}y^3$ occurs on the left-hand side, but there is no term containing y^3 on the right-hand side. $\square$

(6.3.5) Proposition. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let $I = (x^2, y-cx, z^2)$, $c \in K\backslash\{0\}$. Then $I^* = I^F = I$.*

Proof. The socle mod I is xz. Using z as a test element, it suffices to see that $z(xz)^p \notin (x^{2p}, y^p - c^px^p, z^{2p})$. Let $p = 3h+2$ and $f = x^3+y^3$. Suppose $z(xz)^p \in (x^{2p}, y^p - c^px^p, z^{2p})$. This is equivalent to having

$$x^pz^{3h+3} \in (x^{2p}, y^p - c^px^p, z^{6h+4})$$

or

$$x^p(z^3)^{h+1} \in (x^{2p}, y^p - c^px^p, (z^3)^{2h+1}z).$$

Using the basic relation in R we see that this is equivalent to having

$$x^pf^{h+1} \in (x^{2p}, y^p - c^px^p, f^{2h+1}z).$$

Next we use the $\mathbb{Z}_3$-grading (Lemma 4.4.2) to see that this is equivalent to having

$$x^pf^{h+1} \in (x^{2p}, y^p - c^px^p, f^{2h+2})$$

in $K[[x, y]]$. The degree of $x^p f^{h+1}$ is $2p + 1$, while the degree of f^{2h+2} is $2p + 2$. Since we are in the homogeneous case, we may conclude that

$$x^p f^{h+1} = (a_1 x + a_2 y)x^{2p} + B(y^p - c^p x^p)$$

where a_1, $a_2 \in K$ and $B \in K[x, y]$ (6.3.3). Since $x^p f^{h+1}$ has no term with the degree of x less than p, we can conclude that $B = (b_1 x^{p+1} + b_2 x^p y)$, b_1, $b_2 \in K$, and thus

$$x^p f^{h+1} = (a_1 x + a_2 y)x^{2p} + (b_1 x^{p+1} + b_2 x^p y)(y^p - c^p x^p).$$

Expanding $x^p f^{h+1}$ gives

$$(\cdots + (h + 1)x^{2p-2} y^3 + \cdots) = (a_1 x + a_2 y)x^{2p} + (b_1 x^{p+1} + b_2 x^p y)(y^p - c^p x^p).$$

Now it is clear that nothing cancels the term $(h + 1)x^{2p-2} y^3$ on the left-hand side of the equation, so the equality cannot hold. $\square$

(6.3.6) Proposition. *Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let $I = (x^k, y + x, z^2)$ with $k \geq 3$. Then $I^* = I^F = I$.*

Proof. The socle mod I is $x^{k-1}z$. Using z as a test element, it suffices to see that $z(x^{k-1}z)^p \notin (x^{kp}, y^p + x^p, z^{2p})$. Let $p = 3h + 2$ and $f = x^3 + y^3$. Suppose $z(x^{k-1}z)^p \in (x^{kp}, y^p + x^p, z^{2p})$. Then

$$x^{(k-1)p} z^{3h+3} \in (x^{kp}, y^p + x^p, z^{6h+4}),$$

which implies that

$$x^{(k-1)p}(z^3)^{h+1} \in (x^{kp}, y^p + x^p, (z^3)^{2h+1} z).$$

Using the basic relation in R we see that

$$x^{(k-1)p}f^{h+1} \in (x^{kp}, y^p + x^p, f^{2h+1}z).$$

Now we use the $\mathbb{Z}_3$-grading (Lemma 4.4.2) to see that this is equivalent to

$$x^{(k-1)p}f^{h+1} \in (x^{kp}, y^p + x^p, f^{2h+2}).$$

Since we are in the homogeneous case, we may conclude that

$$x^{(k-1)p}f^{h+1} = (a_1x + a_2y)x^{kp} + B(x^p + y^p) + Cf^{2h+2}$$

where $a_1, a_2 \in K$ and B, $C \in K[x, y]$ (6.3.3). Let $x^3 + y^3 = (x + y)Q$, where Q is the quadratic form $x^2 - xy + y^2$. It is clear that $a_1 = a_2$ since $(x + y)$ must divide the term $(a_1x + a_2y)x^{kp}$. So

$$x^{(k-1)p}(x + y)^{h+1}Q^{h+1} = a(x + y)x^{kp} + B(x + y)^p + C(x + y)^{2h+2}Q^{2h+1}.$$

Dividing by $(x + y)$ yields

$$x^{(k-1)p}(x + y)^hQ^{h+1} = ax^{kp} + B(x + y)^{p-1} + C(x + y)^{2h+1}Q^{2h+1}.$$

This implies that $(x + y)^h$ divides ax^{kp} which is clearly false. □

(6.3.7) Proposition. *Let $R = K[[x, y, z]]/(x^3+y^3+z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let $I = (x^k, x+y-dx^{k-1}, z^2)$, $k \geq 3$, $d \in K \setminus\{0\}$. Then $I^* = I^F = I$.*

Proof. The socle mod I is $x^{k-1}z$. Using z as a test element, it suffices to show that $zx^{(k-1)p}z^p \notin (x^{kp}, x^p + y^p - d^px^{(k-1)p}, z^{2p})$. We will reduce to the case $d = 1$.

Apply the following map to R: $x \to \lambda x$, $y \to \lambda y$, and $z \to \lambda z$, where $\lambda \in K$. Then $zx^{k-1}z \in (x^k, x+y-dx^{k-1}, z^2)^*$ if and only if $\lambda^k zx^{k-1} \in (\lambda^k x^k, \lambda x + \lambda y - \lambda^{k-1}dx^{k-1}, \lambda^2 z^2)^*$. By factoring out the λs, we are left with $zx^{k-1} \in (x^k, x+y-\lambda^{k-2}dx^{k-1}, z^2)^*$. If $d \neq 0$, let $\lambda = d^{-1/(k-2)}$. So if $x^{k-1}z$ is in the tight closure of the ideal for one value of $d \neq 0$, then it is in for all $d \neq 0$. We have reduced to the case where $I = (x^k, x+y-x^{k-1}, z^2)$. By Lemma 6.1.2 it is enough to find an ideal $J \subseteq I$ such that J is tightly closed and $R/I \hookrightarrow R/J$. Let $J_0 = ((x+y)^2, x^{2k-2}, (x+y)x^k, x^{2k-1}, (x+y)z^2, x^{k-1}z^2)$. The desired J is ${J_0}^*$. In order to show that $R/I \hookrightarrow R/J_0^*$, it is sufficient to find $v \in J_0^* \colon I$ such that $vu \notin J_0^*$ where v is the socle mod I (Lemma 6.1.2).

First we want to see that ${J_0}^* \subseteq I$. Let $J_1 = (y(x+y), x(x+y), x^k, z^2)$. The socle mod J_1 is generated by $(x+y)z$ and $x^{k-1}z$. We would like to show that $J_1 = {J_1}^*$. We know that $(x+y)z \notin {J_1}^*$ by a degree argument [S4, Theorem 2.2]. To show that $x^{k-1}z \notin {J_1}^*$ we will consider the ideal $J_2 = (x+y, x^k, z^2)$. We know that $x^{k-1}z \notin {J_2}^*$ and $J_2 = {J_2}^*$ by a previous case (Proposition 6.3.6). As $J_1 \subseteq J_2$ and $x^{k-1}z \notin {J_2}^*$, we may conclude that $x^{k-1}z \notin {J_1}^*$. Thus $J_1 = {J_1}^*$. We also know that $J_0 \subseteq J_1$ implies ${J_0}^* \subseteq {J_1}^*$ (Proposition 2.2.3(c)). Now we have ${J_0}^* \subseteq {J_1}^* = J_1 \subseteq I$, which guarantees that ${J_0}^* \subseteq I$.

Next we would like to show that $x+y+x^{k-1} \in {J_0}^* \colon I$. First we note that

$$(x+y+x^{k-1})I \subseteq ((x+y)x^k, x^{2k-1}, (x+y)^2 - x^{2k-2}, (x+y)z^2, x^{k-1}z^2).$$

Certainly

$$\begin{aligned}
&((x+y)x^k, x^{2k-1}, (x+y)^2 - x^{2k-2}, (x+y)z^2, x^{k-1}z^2) \\
&\qquad \subseteq ((x+y)^2, x^{2k-2}, (x+y)x^k, x^{2k-1}, (x+y)z^2, x^{k-1}z^2) = J_0 \subseteq J_0^*.
\end{aligned}$$

Recall that $x^{k-1}z$ is the socle mod I. We want to show that $(x+y+x^{k-1})x^{k-1}z \notin {J_0}^*$. Since J_0 and hence J_0^* is homogeneous, it is enough to show that $(x+y)x^{k-1}z \notin {J_0}^*$. Using z as a test element, it suffices to see that $z(x+y)^p x^{(k-1)p} z^p \notin {J_0}^{[p]}$. Suppose $z(x+y)^p x^{(k-1)p} z^p \in {J_0}^{[p]}$. Then

$$z(x+y)^p x^{(k-1)p} z^p \in \big((x+y)^p x^{kp}, x^{(2k-2)p}, (x+y)^{2p}, (x+y)^p z^{2p}, x^{(k-1)p} z^{2p}\big)$$

and

$$\begin{aligned}
&(x+y)^p x^{(k-1)p} (z^3)^{h+1} \\
&\qquad \in \big((x+y)^p x^{kp}, x^{(2k-2)p}, (x+y)^{2p}, (x+y)^p {(z^3)}^{h+1} z, x^{(k-1)p} {(z^3)}^{h+1} z\big).
\end{aligned}$$

Using the basic relation in R we see that

$$(x+y)^p x^{(k-1)p} f^{h+1} \in \big((x+y)^p x^{kp}, x^{(2k-2)p}, (x+y)^{2p}, (x+y)^p f^{h+1} z, x^{(k-1)p} f^{h+1} z\big).$$

Using the $\mathbb{Z}_3$-grading (Lemma 4.4.2) we see that this is equivalent to having

$$(x+y)^p x^{(k-1)p} f^{h+1} \in ((x+y)^p x^{kp}, x^{(2k-2)p}, (x+y)^{2p}, (x+y)^p f^{h+2}, x^{(k-1)p} f^{h+2}).$$

The degree of $(x+y)^p x^{(k-1)p} f^{h+1}$ is $(k+1)p+1$, while the degree of $x^{(k-1)p} f^{h+2}$ is $(k+1)p+2$. Also, the degree of $x^{(2k-2)p}$ is greater than $(k+1)p+1$ as long as $k > 3$. Since we are in the homogeneous case, we may conclude that

$$(x+y)^p x^{(k-1)p} f^{h+1} \in \big((x+y)^p x^{kp}, (x+y)^{2p}, (x+y)^p f^{h+2}\big).$$

Dividing by $(x+y)^p$ yields

$$x^{(k-1)p} f^{h+1} \in \big(x^{kp}, (x+y)^p, f^{h+2}\big).$$

But this is equivalent to having

$$x^{k-1}z \in \big(x^k, (x+y), z^2\big)^*.$$

We know that $x^{k-1}z \notin (x^k, x+y, z^2)^*$ by a previous result (Proposition 6.3.6).

Let $k=3$ and suppose that

$$(x+y)^p x^{2p} f^{h+1} \in \big((x+y)^p x^{3p}, x^{4p}, (x+y)^{2p}, (x+y)^p f^{h+2}, x^{2p} f^{h+2}\big).$$

The degree of $(x+y)^p x^{2p} f^{h+1}$ is $4p+1$. Since we are in the homogeneous case, this implies that

$$\begin{aligned}(x+y)^p x^{2p} f^{h+1} = A(x+y)^p x^{3p} &+ (\beta_1 x + \beta_2 y)x^{4p} \\ &+ C(x+y)^{2p} + D(x+y)^p f^{h+2} + E x^{2p} f^{h+2}\end{aligned}$$

where $\beta_1, \beta_2 \in K$ and A, C, D and $E \in K[x,y]$ (6.3.3). But this implies that $(x+y)^{h+2}$ divides $(\beta_1 x + \beta_2 y)x^{4p}$ which is impossible.

So with $v = (x+y+x^{k-1})$, we have $v \in J_0^* \colon I$ and $x^{k-1}zv \notin J_0^*$. This is enough to show $R/I \hookrightarrow R/J_0^*$ by Lemma 6.1.2. Since J_0^* is tightly closed, we know that I is tightly closed, also by Lemma 6.1.2. $\square$

In addition to the cases where $I \subseteq (x,y)$, we can determine whether or not an irreducible ideal is tightly closed, not just that $I^* = I^F$, in the following cases.

(6.3.8) Proposition. *Let* $R = K[[x,y,z]]/(x^3+y^3+z^3)$*, where* K *is a field of characteristic* p*. Let* I *be an irreducible* $\mathbb{Z}_3$*-graded ideal of the form* $(H, H\colon f, H\colon f)$*,* *where $f = x^3 + y^3 \in H$, and H is generated by elements whose leading forms are relatively prime quadratic forms. Then $I = I^*$.*

Proof. I is of the form $(Q_1 + C_1, Q_2 + C_2, z)$. Here we mean the ideal generated by $Q_1 + C_1$, $Q_2 + C_2$, and z, not a triple of ideals. Let Q_3 be the third independent quadratic form. By considering the associated graded ring we can see that $K[[x, y]]/(Q_1 + C_1, Q_2 + C_2)$ has dimension four over K, and it follows that $1, x, y, Q_3$ give a basis. Everything of degree three or more will be in H and Q_3 will represent the socle mod I. This also guarantees that $f \in H$. We would like to show that $Q_3 \notin (Q_1 + C_1, Q_2 + C_2, z)^*$. Using the grading and x as a test element, it is sufficient to show that $xQ_3^p \notin (Q_1^p, Q_2^p, f^{h+1})$. This is equivalent to showing that $xQ_3^p + L_1Q_1^p + L_2Q_2^p$ is not divisible by f^{h+1} where L_1 and L_2 are linear forms. We will dehomogenize the equation by setting $y = 1$. If $xQ_3^p + L_1Q_1^p + L_2Q_2^p$ is divisible by f^{h+1}, then $x\overline{Q_3}^p + \overline{L_1Q_1}^p + \overline{L_2Q_2}^p$ is divisible by $\bar{f}^{h+1}$. This implies the derivative with respect to x is divisible by $\bar{f}^h$. Using the fact that we are in characteristic p, we see that the derivative is $\overline{Q_3}^p + \overline{L_1}'\overline{Q_1}^p + \overline{L_2}'\overline{Q_2}^p$. So we need that

$$(\overline{Q_3} + (\overline{L_1}')^{1/p}\overline{Q_1} + (\overline{L_2}')^{1/p}\overline{Q_2})^p$$

is divisible by $\bar{f}^h$. If we rewrite $\bar{f}^h$ as $(x-1)^h(x-\omega)^h(x-\bar{\omega})^h$, we conclude that all three linear factors of $\bar{f}$ divide

$$(\overline{Q_3} + (\overline{L_1}')^{1/p}\overline{Q_1} + (\overline{L_2}')^{1/p}\overline{Q_2}).$$

Since $\overline{Q_1}$ and $\overline{Q_2}$ are still independent over K, this cannot happen. $\square$

(6.3.9) Comment. Let (H, z) and (H, z^2) be two irreducible m-primary ideals of $K[[x, y, z]]/$

$(x^3 + y^3 + z^3)$. Since $(H, z^2) \subseteq (H, z)$, we know that if (H, z^2) is tightly closed, then so is (H, z) (Proposition 6.1.1). In particular, if $I = (x, y, z^2)$, $(x^2, y - cx, z^2)$, $(x^k, y + x, z^2)$, or $(x^k, x + y - x^{k-1}, z^2)$, we know that $I = I^*$. So if $I = (x, y, z)$, $(x^2, y - cx, z)$, $(x^k, y + x, z)$, or $(x^k, x + y - x^{k-1}, z)$, we know that $I = I^*$ also.

The following lemma is used to prove a proposition which gives a useful way to write m-primary ideals in a Gorenstein local ring with a given socle dimension. We will use this characterization to give an alternate proof that $I^* \neq I$ in the case where I is expanded from $K[[x, y]]$.

(6.3.10) Lemma. *Let (R, m, k) be a Noetherian local ring and M a finite length R-module. Let $u_1, \dots, u_r$ generate the maximal ideal of R. Then the dimension of the socle in M is equal to the minimal number of generators of $M^\vee$. ($^\vee = \operatorname{Hom}(\ , E)$ where E is an injective hull of k.)*

Proof. We have the following map:

$$0 \rightarrow \operatorname{Soc} M \rightarrow M \xrightarrow{[u_1, \dots, u_r]} M^r$$

$$z \mapsto (u_1 z, \dots, u_r z).$$

Dualizing gives

$$0 \leftarrow (\operatorname{Soc} M)^\vee \leftarrow M^\vee \xleftarrow{\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}} (M^\vee)^r$$

$$\sum u_i z_i \leftarrow (z_1, \dots, z_r)$$

So $(\operatorname{Soc} M)^{\vee} \cong M^{\vee}/mM^{\vee}$. The dimension of $M^{\vee}/mM^{\vee}$ is exactly the minimal number of generators of $M^{\vee}$. We also know that the dimension of $(\operatorname{Soc} M)^{\vee}$ equals the dimension of $\operatorname{Soc} M$, since $^{\vee}$ preserves length. □

(6.3.11) Proposition. *Let R be a Gorenstein local ring and $x_1, \dots, x_d$ a system of parameters. Suppose I is an m-primary ideal and the dimension of the socle mod I is k. Then $I = (x_1^t, \dots, x_d^t) \colon (f_1, \dots, f_k)$ for some t and some $f_i \in R/(x_1^t, \dots, x_d^t)$.*

Proof. The dimension of the socle mod I is k, so $(R/I)^{\vee} \cong (Rf_1 + \cdots + Rf_k)$ (Lemma 6.3.10). Let E be an injective hull of the residue field for R. Since $(R/I)^{\vee} \cong \operatorname{Hom}(R/I, E) \cong \operatorname{Ann}_E I \subseteq E$, we have that $(Rf_1 + \cdots + Rf_k) \subseteq E$. As R is Gorenstein, we have that $E \cong \underline{lim}_t R/(x_1^t, \dots, x_d^t)$. Thus $(Rf_1 + \cdots + Rf_k) \subseteq R/(x_1^t, \dots, x_d^t)$ for some t. Since R/I is Artin local and hence complete, we know that $(R/I)^{\vee\vee} \cong R/I$. Thus

$$\begin{aligned} R/I &\cong (Rf_1 + \cdots + Rf_k)^{\vee} \\ &\cong \operatorname{Hom}_R((Rf_1 + \cdots + Rf_k), E), \end{aligned}$$

which implies that

$$\begin{aligned} I &\cong \operatorname{Ann}_E(Rf_1 + \cdots + Rf_k) \\ &\cong \operatorname{Ann}_{R/(x_1^t, \dots, x_d^t)}(Rf_1 + \cdots + Rf_k) \\ &\cong (x_1^t, \dots, x_d^t) \colon (f_1, \dots, f_k). \quad \square \end{aligned}$$

(6.3.12) Proposition. *Let $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, where K is a field of characteristic p and $p \equiv 2 \mod 3$. Let I be a $\mathbb{Z}_3$-graded irreducible ideal in R of* *the form (H, H, H), and let u represent the socle mod H in $K[[x, y]]$. Then $I^* = I^F$ and $uz^2 \in I^F$.*

Proof. We know that H is an irreducible m-primary ideal in $K[[x, y]]$. Using Proposition 6.3.11, we can write $H = (x^t, y^t) \colon g$, where $gu \equiv x^{t-1}y^{t-1} \mod (x^t, y^t)$. Using the $\mathbb{Z}_3$-grading (Lemma 4.4.1), we know that uz^2 is the socle mod I. Since I is just H expanded to R, $uz^2 \in I^F$ if and only if $uz^2 \in H^F$. This is equivalent to having $u^q z^{2q} \in H^{[q]}$. Using our characterization of H, we see this is equivalent to

$$u^q z^{2q} \in \big((x^t, y^t) \colon g\big)^{[q]}.$$

We can use the basic relation in R and the $\mathbb{Z}_3$-grading to see that this is equivalent to having

$$u^q f^{2h} \in \big((x^t, y^t) \colon g\big)^{[q]}$$

in $K[[x, y]]$. Now we are working over a regular ring and this is equivalent to

$$u^q f^{2h} \in (x^{qt}, y^{qt}) \colon g^q$$

by Corollary 4.4.8. This is true if and only if

$$u^q g^q f^{2h} = (ug)^q f^{2h} \in (x^{qt}, y^{qt}).$$

Since $g^q u^q \equiv x^{q(t-1)}y^{q(t-1)} \mod (x^{qt}, y^{qt})$, we can write

$$x^{q(t-1)}y^{q(t-1)} f^{2h} \in (x^{qt}, y^{qt}).$$

Dividing by the appropriate powers of x and y gives

$$f^{2h} \in (x^q, y^q),$$

which we know is true by Lemma 4.4.4. □

Next we classify the cases of m-primary irreducible $\mathbb{Z}_3$-graded ideals not included in Theorem 6.3.1. To do this we need the following proposition which gives a characterization of the m-primary irreducible ideals in $K[[x,y]]$.

(6.3.13) Lemma. *Let $A = K[[x,y]]$. Let I be an irreducible m-primary ideal in A. Then I is generated by parameters.*

Proof. First note that I is a height two ideal and the quotient, A/I, is Cohen-Macaulay and has finite projective dimension. This means that A/I must have a resolution that looks like

$$0 \to A^{r-1} \to A^r \to A \to A/I \to 0$$

where the entries of the matrix of the map from A^r to A can be taken to be minimal generators of I. Then I must be the ideal generated by the $r-1$ size minors of of the second matrix. This implies that the type of A/I is one smaller than the number of generators of I. Since A/I has type one, we must have $r = 2$. □

We are now able to classify the remaining cases.

(6.3.14) Proposition. *Let $R = K[[x,y,z]]/(x^3+y^3+z^3)$ and $A = K[[x,y]]$, where K is a field of characteristic $p \neq 3$. Let I be an m-primary irreducible $\mathbb{Z}_3$-graded ideal of R corresponding to the triple of ideals $(H, H, H\colon f)$, where $f = x^3+y^3 \in H$. Suppose H does not contain an element with a linear leading form. Then I has one of the following forms:*

(1) *$I = (Q_1, Q_2, z^2)$ where Q_1 and Q_2 are relatively prime quadratic forms in A*

(2) *$I = ({L_1}^2 + C, L_1L_2 + D, z^2)$ where L_1 and L_2 are independent linear forms, L_1 divides f, and C and D have cubic or higher leading forms*

(3) *$I = (L_1L_2 + C, D, z^2)$ where L_1, L_2, C and D are as in (2)*

Proof. We know that $I = H + Az^2$ where H is an m-primary irreducible ideal of A. Also, we know that H is generated by two parameters by Lemma 6.3.13.

Suppose $H = (Q_1+C_1, Q_2+C_2)$ where Q_1 and Q_2 are quadratic forms and C_1 and C_2 are higher order terms. If Q_1 and Q_2 are relatively prime, then by considering the associated graded ring, we can see that everything of degree three or higher is contained in H. Thus $H = (Q_1, Q_2)$ and the third independent quadratic form will be the socle mod H.

If Q_1 and Q_2 are not relatively prime, we can write $H = (LL_1+C_1, LL_2+C_2)$. If L and L_1 are independent over K, then they span the space of linear forms and we can write $L_2 = aL + bL_1$. This implies that $LL_2 = aL^2 + bLL_1$. Hence we may rewrite H as $(LL_1+C_1, L^2+{C_2}')$. A similar argument applies if L and L_2 are independent. If L, L_1 and L_2 are all dependent, then $H = (L^2 + C_1, L^2 + C_2) = (L^2 + C_1, C_2)$.

If $H = (LL_1 + C_1, L^2 + {C_2}')$, since we must have $f \in H$, either LL_1 divides f or L^2 divides f. Suppose L does not divide f. Then the associated graded ring must contain everything of order three or higher and $f = (L + D_2)(LL_1 + C_1) - (L_1 + D_1)(L^2 + C_2')$. But everything on the right hand side has order three or higher, hence L divides f.

If $H = (L^2 + C_1, C_2)$, then L^2 must divide f. To see this, note that if C_2 divides f, then f will be a minimal generator of H. Since $z^2 \in I$ and $z^3 = -f$, if f is a minimal generator of H, then I will be generated by z^2 and the other minimal generator of H, $L^2 + C_1$. In other words, I will be generated by parameters and we know that the socle mod I is contained in I^F by Proposition 6.1.4. $\square$

(6.3.15) Comment. The remaining cases have proved to be very difficult. In particular, even the question of whether $xyz \in (x^2, y^2, z^2)^*$ remains open. We can show that $xyz \in (x^2, y^2, z^2)^*$ for $p < 200$ and $xyz \in (x^2, y^2, z^2)^F$ for $p \equiv 2 \mod 3$, $p < 200$, but a proof for all p eludes us.

6.4 Generalizations to Other Elliptic Curves

Many of the results in this chapter can be generalized to the elliptic curves $K[[x, y, z]]/$
$(z^3 - F(x, y))$ where $F(x, y)$ is a homogeneous polynomial of degree three.

(6.4.1) Proposition. *Let $R = K[[x, y, z]]/(z^3 - F(x, y))$, where K is a field of characteristic p, $p \equiv 2 \mod 3$, and $F(x, y)$ is a homogeneous polynomial of degree three. Let I be an irreducible m-primary ideal of R with $I \subseteq (x, y)$. Suppose u represents the socle mod I. Then $u \in I^F$.*

Proof. We know that there is an injection $R/(x, y) \hookrightarrow R/I$ (Proposition 6.1.1). It suffices to see that $z^{2p} \in (x^p, y^p)$. Suppose $p = 3h + 2$. Then $z^{2p} = z^{6h+4} = (z^3)^{2h+1}z = F^{2h+1}z$. Now it is enough to see that $F^{2h+1} \in (x^p, y^p)$ in $K[[x, y]]$. The degree of F^{2h+1} is $6h + 3 = 2p - 1$, so every term of F^{2h+1} has a factor of x^p or y^p. In other words, $F^{2h+1} \in (x^p, y^p)$. Again, $z^2 \in (x, y)^F$ implies that $u \in I^F$ (6.1.1). $\square$

CHAPTER 7

REGULAR CLOSURE

In this chapter we discuss regular closure. One of our main motivations for studying regular closure is that it is defined in all characteristics, including mixed characteristic. Also, when an element is in the tight closure of a certain submodule or ideal it is also in the corresponding regular closure. The regular closure contains the tight closure when both are defined, but in general, the regular closure is strictly larger. So, if we could prove that regular closure "captures colons," then we would be able to prove theorems about maps of Tor vanishing in mixed characteristic.

Initially, two different notions of regular closure were discussed in the literature. This was due, in part, to the lack of a good theory of test elements at that time. We would like to determine when these two notions coincide, if at all.

(7.1.1) Definition. Let $N \subseteq M$ be arbitrary modules over a Noetherian ring R. We say that $x \in M$ is in the *regular closure* of N in M, denoted N_M^{REG}, if for every homomorphism of R into a regular ring S which maps R° into S°, the image of x in $S \otimes_R M$ is in the image of $S \otimes_R N$ in $S \otimes_R M$. If we consider all maps $R \to S$ with S regular but without the requirement that R° maps into S°, then we say that $x \in N_M^{\mathrm{reg}}$.

(7.1.2) Comment. Clearly $I^{\text{reg}} \subseteq I^{\text{REG}}$. It is also true that $I^* \subseteq I^{\text{REG}} \subseteq \bar{I}$ whenever tight closure is defined. The first inclusion is immediate from the definition of tight closure and the fact that every ideal in a regular ring is tightly closed. The second inclusion follows from the fact that integral closure is tested by mapping to DVR's. It is also true that $I^* \subseteq I^{\text{reg}}$. We will show that it is sufficient to test regular closure by considering maps to complete regular local rings. If $h\colon R \to S$ is a homomorphism of Noetherian rings of characteristic p, then if S is a complete local ring and $x \in R$ is in the tight closure of $I \subseteq R$, then $h(x) \in (IS)^*$ [HH5]. Since every ideal in a regular ring is tightly closed, we have $I^* \subseteq I^{\text{reg}}$.

(7.1.3) Example. An example which shows that in characteristic p the tight closure can be strictly smaller than the regular closure is $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$. One can show that z is in the regular closure of $(x, y)R$ but not in the tight closure [HH4].

We now discuss some reductions one can make in studying regular closure.

(7.1.4) Proposition. *In testing $u \in I^{\text{reg}}/I^{\text{REG}}$ it suffices to look at local homomorphisms into complete regular local rings.*

Proof. First, it is sufficient to look at maps into regular local rings. Suppose $u \notin I^{\text{reg}}$. So we can give a map $R \to S$ with S regular and $u \notin IS$. Let Q be a prime containing $IS : u$. Then $u \notin IS_Q$. To see this note that if $u \in IS_Q$, then u can be written as is/r where $i \in I, s \in S$ and $r \in S \backslash Q$. If $u = is/r$ is in S_Q, then there exists $d \in S \backslash Q$ with $d(ur - is) = 0$. But this implies that $dur = dis$ and hence $dr \in IS : u$. But d and r are both in $S \backslash Q$.

Next we show that it is sufficient to consider only local homomorphisms into regular local rings. Suppose S is local and $u \notin I^{\text{reg}}$. So there is a map $R \to S$, not necessarily a local homomorphism, with $u \notin IS$. Again, localize at a prime containing $IS : u$, say P. Then $u \notin IS_P$. I maps into $m_{S_P} = PS_P$ since I maps to $IS \subseteq IS : u \subseteq P$. Let $Q = P^c$. So $I \subseteq P^c$. The map $R_{P^c} \to S_P$ is a local homomorphism of local rings. As before, $u/1 \notin IS_P$.

Finally, it is sufficient to look at local homomorphisms into complete regular local rings. Suppose $u \notin I^{\text{reg}}$. So there is a map $R \to S$ with $u \notin IS$. The map $S \to \hat{S}$ is flat and local and hence faithfully flat. So $I\hat{S} \cap S = I$. Let $u \in I\hat{S}$, then $u \in IS$ which is a contradiction.

The property that R° maps into S° is preserved by composition. Also, the property holds if $R \to S$ is flat. As localization and completion are both flat, we can make the same reductions as above when testing $u \in I^{\text{REG}}$. $\square$

(7.1.5) Discussion. When R is essentially of finite type over a perfect field or when R is a finitely generated algebra over a perfect field we can make further reductions in testing regular closure. Let K be a field essentially of finite type over a perfect field K_0. Let R be a Noetherian ring essentially of finite type over K. Let $I \subseteq R$ and $u \in R$ and suppose $u \notin I^{\text{reg}}$. We can already reduce to the case where S is a complete regular local ring by Proposition 7.1.4., so we have a map $R \to S$ with S complete regular local and $u \notin IS$. Since R is essentially of finite type over K, R is also essentially of finite type over K_0. By this we mean that R is a localization of a finitely generated K_0-algebra. As K_0 is perfect, the image of K_0 in S can be enlarged to a coefficient field for S, say $K_0 \subseteq L$. Then $S \cong L[[x_1, \dots, x_n]]$.

Suppose R is a localization of $K_0[\theta_1, \ldots, \theta_n]$. Then we have the following maps

$$K_0[\theta_1, \ldots, \theta_n] \to K_0[v_1, \ldots, v_n] \to L[v_1, \ldots, v_n] \subseteq L[[x_1, \ldots, x_n]],$$

where v_i is the image of θ_i in S. By a theorem of Artin and Rotthaus [ArR], the map $L[v_1, \ldots, v_n] \to L[[x_1, \ldots, x_n]]$ factors through $L[x_1, \ldots, x_m, y_1, \ldots, y_n]^h_{(x,y)}$, where x, y are algebraically independent elements (over L) of the maximal ideal of $L[[x]]$ and ^{h} denotes Henselization. In addition, the map

$$L[x_1, \ldots, x_m, y_1, \ldots, y_n]^h_{(x,y)} \to L[[x_1, \ldots, x_n]]$$

is local. If R is actually an affine K_0-algebra, then instead of taking the entire Henselization, we can take a pointed étale extension of $L[x, y]_{(x,y)}$, and, hence, an étale extension of $L[x, y]$, since a pointed étale extension is a direct limit of étale extensions. By another direct limit argument we may also replace L by a regular affine K_0-algebra, so one may test regular closure in this case by mapping only to affine K_0-algebras.

Since the maximal ideal of $L[x_1, \ldots, x_m, y_1, \ldots, y_n]^h_{(x,y)}$ contracts to the maximal ideal of $K_0[\theta_1, \ldots, \theta_n]$ and R is a localization of $K_0[\theta_1, \ldots, \theta_n]$, the map $K_0[\theta_1, \ldots, \theta_n] \to L[x_1, \ldots, x_m, y_1, \ldots, y_n]^h_{(x,y)}$ factors through R. In other words, we now have a map $R \to L[x_1, \ldots, x_m, y_1, \ldots, y_n]^h_{(x,y)}$ and this is a factorization of the map $R \to L[[x_1, \ldots, x_n]]$. $\square$

Next we show that we can test regular closure in $K[[x, y, z]]/(x^3+y^3+z^3)$ using maps to $\mathbb{Z}_p[A, B, C]_{AB}[[V]]/(A^3+B^3+C^3)$ where A, B, C and V are indeterminates.

(7.1.6) Proposition. *Let* $R = K[[x, y, z]]/(x^3+y^3+z^3)$, $u \in R$ *and* I *an ideal in*

R. Then $u \in I^{\mathrm{reg}}/I^{\mathrm{REG}}$ if and only if $u \in IT_{AB}$ where $T_{AB} = \mathbb{Z}_p[A,B,C]_{AB}[[V]]/(A^3+B^3+C^3)$.

Proof. First note that there is a natural map $R \to T_{AB}$ with $x \mapsto AV$, $y \mapsto BV$ and $z \mapsto CV$. This map factors through any map $R \to S$ where S is a regular local ring. To see this, we note that any map $R \to S$ must be of the form $x \mapsto \alpha d$, $y \mapsto \beta d$, and $z \mapsto \gamma d$ where $d \in S$, $\alpha^3 + \beta^3 + \gamma^3 = 0$ and either at least two of α, β, γ are units or all three elements are 0 [HH4].

To see that T_{AB} is regular, first note that

$$\begin{aligned}\mathbb{Z}_p[A,B,C]_A/(A^3+B^3+C^3) &\cong \mathbb{Z}_p[A,A^{-1},B/A,C/A]_A/(1+(B/A)^3+(C/A)^3)\\ &\cong \mathbb{Z}_p[r,s]/(1+r^3+s^3),\end{aligned}$$

which is certainly regular. So $\mathbb{Z}_p[A,B,C]_{AB}/(A^3+B^3+C^3)$ is regular, as is $\mathbb{Z}_p[A,B,C]_{AB}[[V]]/(A^3+B^3+C^3)$. $\square$

Finally, we show that in certain rings, including $K[[x,y,z]]/(x^3+y^3+z^3)$, $I^{\mathrm{reg}} = I^{\mathrm{REG}}$ for all $I \subset R$.

(7.1.7) Proposition. *Let (R,m) be a local domain. Suppose the image of m in any power series ring is principal and $Bl_m R$, the blowup of m in R, is regular. Then $I^{\mathrm{reg}} = I^{\mathrm{REG}}$ for all $I \subset R$.*

Proof. Let $m = (x_1, \dots, x_n)$. Suppose $u \notin I^{\mathrm{reg}}$. Then there exists a map $R \to S$ with $u \notin IS$. As before, we can assume the map is local and S is a complete regular local ring, i.e. S is a power series ring. Since mS is principal, $f(x_i)$ divides $f(x_j)$ for some i and all j. Let $f(x_j) = a_j f(x_i)$. Next we would like to see that the following diagram commutes and $h(u) \notin IR[x_j/x_i]$:

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "h"'] & S \\
R[x_j/x_i] &
\end{tikzcd}$$

The map $h\colon R \to R[x_j/x_i]$ is the obvious inclusion. Let $g\colon R[x_j/x_i] \to S$ be defined as follows: $g(r) = f(r)$ for $r \in R$ and $g(x_j/x_i) = a_j$. Suppose $h(u) \in IR[x_j/x_i]$. We can write $h(u) = \sum w_k i_k$ where $w_k \in R[x_j/x_i]$ and $i_k \in I$. Then

$$g(h(u)) = g(\sum w_k i_k) = \sum g(w_k)g(i_k) = \sum g(w_k)f(i_k)$$

since $i_k \in R$.

So $g(h(u)) \in IS$. But $g(h(u)) = f(u)$ and $f(u) \notin IS$. This is a contradiction. Since $R[x_i/x_j]$ is regular, it is sufficient to test maps to these rings. $\square$

(7.1.8) Remark. The rings $R = K[[x, y, z]]/(x^3 + y^3 + z^3)$, char $K \neq 3$, satisfy the conditions of Proposition 7.1.7 and so $I^{\text{reg}} = I^{\text{REG}}$ for these rings.

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