MCS-236: Graph Theory Handout #3
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Relations, Functions, and Sequences

Relations

- An *ordered pair* can be constructed from any two mathematical objects. For example, the ordered pair $(2, 1)$ has 2 as its *first component* and 1 as its *second component*. The ordered pair $(0, 0)$ has 0 in both components. If $*$ stands for the multiplication operation then the ordered pair $(\mathbb{N}, *)$ has the set of natural numbers as its first component and multiplication as its second component.

- Two ordered pairs (a, b) and (c, d) are said to be *equal*, written $(a, b) = (c, d)$, if $a = c$ and $b = d$.

- An ordered pair is different from an unordered pair. So $(a, a) \neq \{a, a\}$, $(a, b) \neq (b, a)$, but $\{a, b\} = \{b, a\}$.

- Given two sets A and B, we define its *Cartesian product*, written $A \times B$, to be $A \times B = \{(a, b) : a \in A, b \in B\}$. For example, the plane we study in analytic geometry is simply $\mathbb{R} \times \mathbb{R}$ (also written $\mathbb{R}^2$).

- A *relation R from A to B* is some subset of $A \times B$. If $(a, b) \in R$, we say that *a is related to b in R*, and write aRb.

  The empty set $\emptyset$ is the smallest relation from A to B.

  The relation $A \times B$ is the biggest relation from A to B.

- A relation R from A to A is called a *relation on A*. Here are some examples.

  The unit circle U centered at the origin is a relation on $\mathbb{R}$ since U is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, a subset of $\mathbb{R}^2$.

  The relations $<$, $\leq$, $>$, $\geq$, $=$ and their negations are all relations on $\mathbb{R}$.

  'is a brother (sister, parent, sibling, etc) of' are relations on humans.

- A relation R on A is said to be *reflexive* if aRa for all $a \in A$.
- A relation R on A is said to be *symmetric* if bRa whenever aRb.
- A relation R on A is said to be *transitive* if aRc whenever both aRb and bRc.
- (How to axiomatize equality.) A relation R on a set A is said to be an *equivalence relation* if R is *reflexive*, *symmetric*, and *transitive*.
- Show that if we define "aRb if $a^2 - b^2$ is even", then R is an equivalence relation on $\mathbb{Z}$.
- A partition of a set S is a nonempty collection of disjoint, nonempty subsets of S whose union equals S. For example, if $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ is a partition of S into k subsets, then we have that (i) $S_i \neq \emptyset$ for any i, (ii) $\bigcup_{i=1}^{k} S_i = S$, and (iii) $S_i \cap S_j = \emptyset$ if $i \neq j$. In general, a partition may be infinite.
- Equivalence relation and partition are closely related concepts. Given an equivalence relation, there is a unique partition associated with it, and vice versa.
- Let X be any set and let $\mathcal{C}$ be any collection of subsets of X, i.e., for every C, if $C \in \mathcal{C}$ then $C \subseteq X$. We define $\bigcup \mathcal{C}$ to be $\bigcup \mathcal{C} = \{y : y \in C \text{ for some } C \in \mathcal{C}\}$.

  Let R be an equivalence relation on X. For each $a \in X$, define the *equivalence class of* a under the relation R, written $[a]_R$, to be the set of all elements of X that a is related to, i.e., $[a]_R = \{x \in X : aRx\}$.

  Theorem. *For any $a, b \in X$ and any equivalence relation R on X, we have aRb if and only if $[a]_R = [b]_R$.*

  Proof. Assume that a, b are any elements of X and R is any equivalence relation on X.

  Suppose aRb. We will prove that $[a]_R = [b]_R$. We'll first show that $[a]_R \subseteq [b]_R$. So let c be an arbitrary element in $[a]_R$. By definition of $[a]_R$, we have aRc. So by symmetry of R, we see that cRa. By assumption, aRb. So by transitivity of R, it follows that cRb. Again by symmetry of R, we see that bRc. So by defintion of $[b]_R$, we conclude that $c \in [b]_R$. We have now shown that every elment in $[a]_R$ is also a member of $[b]_R$. In other words, $[a]_R \subseteq [b]_R$.

That $[b]_R \subseteq [a]_R$ can be proved in a similar way. Thus $[a]_R = [b]_R$.

Conversely, suppose that $[a]_R = [b]_R$. We will show that aRb. Since R is reflexive, we know that bRb. So by definition of $[b]_R$, we conclude that $b \in [b]_R$. Since $[a]_R = [b]_R$ by assumption, we then have that $b \in [a]_R$. So by defintion of $[a]_R$, we conclude that aRb. □

Theorem. *Let R be an equivalence relation on X. Define $\mathcal{P}$ to be $\mathcal{P} = \{[x]_R : x \in X\}$. The collection $\mathcal{P}$ is then a partition of X.*

Proof. To prove that $\mathcal{P}$ is a partition of X, we have to show that

1. $[x]_R \neq \emptyset$ for all $x \in X$,
2. for any $a, b \in X$, if $[a]_R \cap [b]_R \neq \emptyset$ then $[a]_R = [b]_R$, and
3. $\bigcup \mathcal{P} = X$.

We prove item 1 as follows. Let x be an arbitrary element of X. By the reflexivity of R, we conclude xRx. This means that the equivalence class $[x]_R$ is not empty since it contains at least an element, specifically element x.

We prove item 2 as follows. Suppose a, b are any elements of X such that $[a]_R \cap [b]_R \neq \emptyset$. Let $c \in ([a]_R \cap [b]_R)$. Then $c \in [a]_R$ and $c \in [b]_R$. Since $c \in [a]_R$, we see that aRc by definition of $[a]_R$. Therefore, $[a]_R = [c]_R$ by last theorem. Since $c \in [b]_R$, we see that bRc by definition of $[b]_R$. Therefore, $[b]_R = [c]_R$ by last theorem. (Therefore, $[c]_R = [b]_R$ as well since equality of sets is an equivalence relation.) Therefore, $[a]_R = [c]_R$ and $[c]_R = [b]_R$. So $[a]_R = [b]_R$ by transitivity of set equality.

We prove item 3 as follows. Since each element of $\mathcal{P}$ is a subset of X, we conclude that $\bigcup \mathcal{P} \subseteq X$. Now let x be an arbitrary element of X. Then $x \in [x]_R \in \mathcal{P}$. Therefore, $x \in \bigcup \mathcal{P}$. Thus, $X \subseteq \bigcup \mathcal{P}$. Hence, $\bigcup \mathcal{P} = X$. □

Theorem. *Let $\mathcal{C}$ be any partition of X. Define R to be a relation on X by declaring that for any $a, b \in X$, it holds that aRb if and only if there exists a $C \in \mathcal{C}$ such that $a \in C$ and $b \in C$. The relation R so defined is then an equivalence relation on X.*

Proof. ... □

- For any relation R from A to B, we define its *inverse relation* R^{-1} to be $\{(b, a) \subseteq B \times A : (a, b) \in R\}$.

- A relation R on X is called *antisymmetric* if for all $a, b \in X$ if aRb and bRa, then $a = b$. A relation R on X is called *asymmetric* if for no $a \in X$ is aRa. A relation R on X is called a *partial order* if it is reflexive, transitive, and antisymmetric. A partial order R on X is a *total order* (or *linear order*) if for all $a, b \in X$, either aRb or bRa. Let R be a partial order on X. An element a in X is called a *minimal* element if for any element $b \in X$, if bRa then $b = a$. Similarly, an element a in X is called a *maximal* element if for any element $b \in X$, if aRb then $b = a$. An element a in X is called a *minimum* element if aRb for all elements $b \in X$. An element a in X is called a *maximum* element if bRa for all elements $b \in X$.

- **Exercises:**

  1. Let S be a nonempty set, and let $\mathcal{C}$ be the collection of all subsets of S, i.e., $\mathcal{C} = \{C : C \subseteq S\}$. Prove that $\subseteq$ is a partial order on $\mathcal{C}$.

     Show that if $|S| > 1$, then $\subseteq$ is not a a total order.

  2. Let $X = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define R to be a relation on X by declaring that for all $a, b \in X$, we have aRb if and only if a is factor of b. For example, $2R8$, $3R9$, but it's not true that $9R8$. Prove that R is a partial order on X.

     Show that R is not a total order.

  3. Prove that $\leq$ is a total order on $\mathbb{R}$.

  4. For each of the partial orders listed above, list all elements that are minimal, maximal, minimum, or maximum.

Functions

- A function f from set A to set B, written $f : A \to B$, is a relation from A to B such that each $a \in A$ is the first component of exactly one ordered pair. Set A is called the *domain* and set B is called the *codomain* of f. If $(a, b) \in f$, we write $b = f(a)$ and call b the *image of a under f*.

  The *range of f* is defined to be $\{b \in B : b = f(a) \text{ for some } a\}$.

- A function f is *injective (1-to-1)* if $f(a) = f(a') \implies a = a'$ for all a, a'.

  A function is *surjective (onto)* if its range equals its codomain.

  A function is *bijective (1-to-1 and onto)* if it is both injective and surjective.

- **Theorem.** A function $f : A \to A$ that is 1-1 (onto) is not necessarily onto (1-1) unless A is finite.

  Proof. . . . □

- Given functions $f : A \to B$ and $g : B \to C$ we define the *composite function* $g \circ f$ to be the function from A to C such that $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

- **Theorem 2.4** (Appendix 2, CZ). If $f : A \to B$ and $g : B \to C$ are bijective, then $g \circ f$ is bijective.

  Proof. . . . □

- **Theorem 2.5** (Appendix 2, CZ). If $f : A \to B$ is a function, then f^{-1} is a bijective function if and only if f is bijective.

  Proof. . . . □

- A permutation of A is any bijective $f : A \to A$.

- $|A| = |B|$ if there is a bijection $f : A \to B$

Sequences

- Let S be any nonempty set. A infinite sequence in S is a function from $\mathbb{N}$ to S. A finite sequence in S is a function from $\{0, 1, 2, \ldots, n\}$ to S for some $n \in \mathbb{N}$.
- We usually write a sequence by enumerating its range like so

  $$\langle a_0, a_1, a_2, \ldots \rangle$$

  or like so

  $$(s_0, s_1, s_2, s_3, s_4, s_5)$$

  or even like so

  $$v_0, v_1, v_2, v_3, v_4.$$

- We sometimes start counting from 1 instead of 0. So it's common to also see a sequence written as

  $$v_1, v_2, v_3, v_4, v_5$$

  etc.