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Basic Set Theory

- A set is a collection of objects considered as a whole. The main concept is that of *membership*. Given an object x and a set A, we should be able to answer whether x is a member of A or not.

- Similar terms: *set, class, collection, family, space*

- *finite sets* vs. *infinite sets*

- Some important number sets

  $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

  $\mathbb{Z}$ = the set of all integers

  $\mathbb{Q}$ = the set of all rational numbers

  $\mathbb{R}$ = the set of all real numbers

  $\mathbb{C}$ = the set of all complex numbers

- Examples of sets written

  by enumeration: $\{1, 2, 3\}$

  by set former: $\{n \in \mathbb{Z} : |n| \leq 3\}$

  by formulaic set former: $\{n^2 : n \in \mathbb{N}\}$

- The empty set $\emptyset$ (sometimes written $\{\}$) has no member.

- Two sets are equal, written $A = B$, if they contain the same elements.

- Notation:

$\in$	is a member of	$\notin$	is not a member of
$\subseteq$	is a subset of	$\subset$	is a proper subset of
$\not\subseteq$	is not a subset of	$\not\subset$	is not a proper subset of
$\supseteq$	is a superset of	$\supset$	is a proper superset of
$\not\supseteq$	is not a superset of	$\not\supset$	is not a proper superset of

where

$A \subseteq B$ means "for all x, if $x \in A$ then $x \in B$."

$A \subset B$ means "$A \subseteq B$ and $A \neq B$."

$A \supseteq B$ means "$B \subseteq A$."

$A \supset B$ means "$B \subset A$."

- *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*
- Set operations:

$\cup$	union
$\cap$	intersection
$\setminus$	set difference
$\overline{A}$	complement of A

where

$A \cup B$ means $\{\, x : x \in A$ or $x \in B \,\}$.

$A \cap B$ means $\{\, x : x \in A$ and $x \in B \,\}$.

$A \setminus B$ means $\{\, x : x \in A$ and $x \notin B \,\}$.

$\overline{A}$ means $U \setminus A$ (where U is the "universal set").

- Sets A and B are *disjoint* if their intersection is empty, i.e., $A \cap B = \emptyset$.
- Venn-Euler Diagram can help one understand why some theorem is true, or even suggest a proof.
- Some rules governing set operations:

$A \cup B = B \cup A$	(commutativity of union)
$A \cap B = B \cap A$	(commutativity of intersection)
$(A \cup B) \cup C = A \cup (B \cup C)$	(associativity of union)
$(A \cap B) \cap C = A \cap (B \cap C)$	(associativity of intersection)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(distributivity)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(distributivity)
$\overline{\overline{A}} = A$	(double complement)
$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$	(DeMorgan's laws)

- Definition of $+$: $|A| + |B| = |A \cup B|$ whenever $A \cap B = \emptyset$.

  Theorem. $|A \cup B| = |A| + |B| - |A \cap B|$

  Proof. . . . □

- min, max, $\sum$, $\prod$, $\bigcup$, $\bigcap$: definition, notation

- work through an example of min, max combination of a real matrix

- $\min(A \cup B) = \min\{\min A, \min B\}$ $\qquad \min \bigcup_{i \in I} A_i = \min\{\min A_i : i \in I\}$

  similarly for max

- $\min \emptyset = +\infty \qquad \max \emptyset = -\infty$

- The power set $\mathcal{P}(A)$ (or 2^A) of A is the collection of all subsets of A. In other words, $\mathcal{P}(A) = \{S : S \subseteq A\}$.

  Theorem. *If A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.*

  Proof. . . . □

- Given a set of n objects, the number of ways to select k objects from them is written $\binom{n}{k}$, and is read n *choose* k.

  Theorem. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

  Proof. . . . □

  In particular, $\binom{n}{2} = \frac{n(n-1)}{2}$.

Theorem (Pigeonhole Principle, Version 1)**.** *If $k+1$ pigeons fly into k holes, then at least 1 hole has at least 2 pigeons.*

Proof. ... $\square$

Theorem (Pigeonhole Principle, Version 2)**.** *If $nk+1$ pigeons fly into k holes, then at least 1 hole has at least $n+1$ pigeons.*

Proof. ... $\square$

Theorem (Pigeonhole Principle, CZ Version)**.** *If n pigeons fly into k holes, then at least 1 hole has at least $\lceil n/k \rceil$ pigeons.*

Proof. ... $\square$

Theorem (Ramsey's Theorem)**.** *Let $P = \{S_1, S_2, \ldots, S_k\}$ be a partition of a set S into k subsets, and let $n_1, n_2, \ldots, n_k$ be k positive integers such that $|S_i| \geq n_i$ for every integer i with $1 \leq i \leq k$. Then there exists a positive integer N such that every N-element subset of S contains at least n_i elements of S_i for some $i (1 \leq i \leq k)$.*

In particular, the integer

$$N = 1 + \sum_{i=1}^{k} (n_i - 1)$$

is the least integer with this property.

Proof. ... $\square$