San Skulrattanakulchai
Gustavus Adolphus College

Logic

- A *proposition* is a statement that is either *true* or *false*, but not both.
- Example propositions:
  - $\sqrt{2}$ is an irrational number.
  - $1 < 1$.
  - Some integers x, y, z satisfy $x^3 + y^3 = z^3$.
  - There are no positive integers x, y, z such that $x^4 + y^4 = z^4$.
- Example non-propositions:
  - What do you mean by that?
  - How pretty!
  - Please give me a call.
  - Barack Obama is an honest president.
  - 23 is an interesting number.
  - $x^2 + 3x - 4 = 0$.
  - This statement is false.

- A *compound proposition* can be constructed from any propositions p, q using the *logical connectives* $\wedge$, $\vee$, $\neg$, $\implies$, $\iff$.

Compound Proposition	**Formal Sentence**	**Meaning**
conjunction	$p \wedge q$	p and q
disjunction	$p \vee q$	p or q
negation	$\neg p$	not p
implication	$p \implies q$	p implies q
biconditional	$p \iff q$	p if and only if q

- In the implication $p \implies q$, the proposition p is called *premise* or *hypothesis* or *antecedent*, and the proposition q is called *conclusion* or *consequence* or *consequent*.

- Truth Tables

  ...

- These are some of the ways to express the implication "$p \implies q$" in English.

  if p **then** q

  p **implies** q

  q **is implied by** p

  p **only if** q

  q **if** p

  p **is sufficient for** q

  q **is necessary for** p

  q **whenever** p

- These are some of the ways to express the biconditional "$p \iff q$" in English.

  p **if and only if** q

  p **is necessary and sufficient for** q

  q **is necessary and sufficient for** p

  p **implies** q **and vice versa**

  if p **then** q **and vice versa**

  $\ldots$

- Definitions:
  - An implication is said to be *trivially true* when its conclusion is true.
  - An implication is said to be *vacuously true* when its hypothesis is false.
  - The *converse* of "$p \implies q$" is "$q \implies p$."
  - The *contrapositive* of "$p \implies q$" is "$\neg q \implies \neg p$."
  - Two propositions are *logically equivalent* when they have the same truth value for all possible combinations of the truth values of their constituent propositions.
  - A *tautology* is a proposition that's always true, eg, "$p \vee \neg p$."
  - A *contradiction* is a proposition that's always false, eg, "$p \wedge \neg p$."
  - A *contingency* is proposition that's sometimes true and sometimes false, eg, "$p \vee p$."
- Question: which of these are tautology? contradiction? contingency?

  $p \implies p$

  $(\neg p) \implies p$

  $[(p \implies q) \wedge p] \wedge (\neg q)$

  $[p \vee (p \wedge q)] \implies p$

  $[p \wedge (p \vee q)] \iff \neg p$

- Some important logical equivalences

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge T \equiv p$	$p \vee F \equiv p$
$p \wedge F \equiv F$	$p \vee T \equiv T$
$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$	$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$
$\neg(\neg p) \equiv p$	$p \iff q \equiv q \iff p$
$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
$p \implies q \equiv (\neg q) \implies (\neg p)$	$p \implies q \equiv (\neg p) \vee q$
$\neg(p \implies q) \equiv p \wedge (\neg q)$	$(\neg p) \implies F \equiv p$
$p \iff q \equiv (p \implies q) \wedge (q \implies p)$	$p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$(p \vee q) \implies r \equiv (p \implies r) \wedge (q \implies r)$	

- Non-proposition statements containing variables, the so-called *open sentences*, can be turned into propositions by quantifying the variables.
- The *existential quantifier* $\exists$ means "at least one." In English, we can express it like

  there is ... there exists ... some ... for some ...

- The *universal quantifier* $\forall$ means "all." Some ways to say it in English are

all ...	for all ...	every ...	for every ...
each ...	for each ...	any ...	for any ...

- Question: What do the following English sentences mean?
  - There is a male student in the MCS-236 class.
  - There is a female student in the MCS-236 class.
  - A rational number is not integral.
  - Odd integers are multiples of 3.
  - Some schiffs are fibs.
  - All fibs are fobs.
  - $a + b = b + a$.
- Exercise: Rewrite all ambiguous sentences above to disambiguate them.

- Here are two common conventions in mathematics writing.
  - Universal quantifiers are routinely omitted when possible.
  - In definition, the word 'if' actually means 'if and only if.'
- Beware of the following.
  - Depending on context, the word 'one' could mean 'at least one' or 'exactly one.' Similarly for 'two', 'there', ..., etc.
  - Oftentimes the word 'a' should be replaced by 'any.'
- To negate quantified propositions, use the logical equivalence

  $$\neg[\forall x.\varphi(x)] \equiv \exists x.\neg\varphi(x)$$

  or the logical equivalence

  $$\neg[\exists x.\varphi(x)] \equiv \forall x.\neg\varphi(x)$$

- A sentence of the form $\forall x \in X.\varphi(x)$ is short for $\forall x[x \in X \implies \varphi(x)]$.
- A sentence of the form $\exists x \in X.\varphi(x)$ is short for $\exists x[x \in X \wedge \varphi(x)]$.
- A sentence of the form, "For every element of the nonempty set ... blah ..." is always true.
- A sentence of the form, "There exists some element in the nonempty set such that ... blah ..." is always false.
- The term *unique* in mathematical writing means "exactly one."
- The term *distinct* in mathematical writing means "different."
- The term *write* in mathematical writing often means "let."
- The phrase *without loss of generality* in mathematical writing has many meanings, as will be demonstrated throughout this course!