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# Relations, Functions, and Sequences

## Relations

- An *ordered pair* can be constructed from any two mathematical objects. For example, the ordered pair $(2, 1)$ has 2 as its *first component* and 1 as its *second component*. The ordered pair $(0, 0)$ has 0 in both components. If $\cdot$ stands for the multiplication operation then the ordered pair $(\mathbb{N}, \cdot)$ has the set of natural numbers as its first component and multiplication as its second component.

- Two ordered pairs $(a, b)$ and $(c, d)$ are said to be *equal*, written $(a, b) = (c, d)$, if $a = c$ and $b = d$.

- An ordered pair is different from an unordered pair. So $(a, a) \neq \{a, a\}$, $(a, b) \neq (b, a)$, but $\{a, b\} = \{b, a\}$.

- Given two sets $A$ and $B$, we define its *Cartesian product*, written $A \times B$, to be $A \times B = \{(a, b) : a \in A, b \in B\}$. For example, the plane we study in analytic geometry is simply $\mathbb{R} \times \mathbb{R}$ (also written $\mathbb{R}^2$).

- A *relation $R$ from $A$ to $B$* is some subset of $A \times B$. If $(a, b) \in R$, we say that *$a$ is related to $b$ in $R$*, and write $aRb$.

  The empty set $\emptyset$ is the smallest relation from $A$ to $B$.

  The relation $A \times B$ is the biggest relation from $A$ to $B$.

- A relation $R$ from $A$ to $A$ is called a *relation on $A$*. Here are some examples.

  The unit circle $U$ centered at the origin is a relation on $\mathbb{R}$ since $U$ is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, a subset of $\mathbb{R}^2$.

  The relations $<$, $\leq$, $>$, $\geq$, $=$ and their negations are all relations on $\mathbb{R}$.

  'is a brother (sister, parent, sibling, etc) of' are relations on humans.

- A relation $R$ on $A$ is said to be *reflexive* if $aRa$ for all $a \in A$.
- A relation $R$ on $A$ is said to be *symmetric* if $bRa$ whenever $aRb$.
- A relation $R$ on $A$ is said to be *transitive* if $aRc$ whenever both $aRb$ and $bRc$.
- (How to axiomatize equality.) A relation $R$ on a set $A$ is said to be an *equivalence relation* if $R$ is *reflexive*, *symmetric*, and *transitive*.
- Show that if we define "$aRb$ if $a^2 - b^2$ is even", then $R$ is an equivalence relation on $\mathbb{Z}$.
- A partition of a set $S$ is a nonempty collection of disjoint, nonempty subsets of $S$ whose union equals $S$. For example, if $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ is a partition of $S$ into $k$ subsets, then we have that (i) $S_i \neq \emptyset$ for any $i$, (ii) $\bigcup_{i=1}^{k} S_i = S$, and (iii) $S_i \cap S_j = \emptyset$ if $i \neq j$. In general, a partition may be infinite.
- Equivalence relation and partition are closely related concepts. Given an equivalence relation, there is a unique partition associated with it, and vice versa.
- Let $X$ be any set and let $\mathcal{C}$ be any collection of subsets of $X$, i.e., for every $C$, if $C \in \mathcal{C}$ then $C \subseteq X$. We define $\bigcup \mathcal{C}$ to be $\bigcup \mathcal{C} = \{y : y \in C \text{ for some } C \in \mathcal{C}\}$.

  Let $R$ be an equivalence relation on $X$. For each $a \in X$, define the *equivalence class of* $a$ under the relation $R$, written $[a]_R$, to be the set of all elements of $X$ that $a$ is related to, i.e., $[a]_R = \{x \in X : aRx\}$.

  **Theorem.** *For any $a, b \in X$ and any equivalence relation $R$ on $X$, we have $aRb$ if and only if $[a]_R = [b]_R$.*

  *Proof.* Assume that $a$, $b$ are any elements of $X$ and $R$ is any equivalence relation on $X$.

  Suppose $aRb$. We will prove that $[a]_R = [b]_R$. We'll first show that $[a]_R \subseteq [b]_R$. So let $c$ be an arbitrary element in $[a]_R$. By definition of $[a]_R$, we have $aRc$. So by symmetry of $R$, we see that $cRa$. By assumption, $aRb$. So by transitivity of $R$, it follows that $cRb$. Again by symmetry of $R$, we see that $bRc$. So by defintion of $[b]_R$, we conclude that $c \in [b]_R$. We have now shown that every elment in $[a]_R$ is also a member of $[b]_R$. In other words, $[a]_R \subseteq [b]_R$.

That $[b]_R \subseteq [a]_R$ can be proved in a similar way. Thus $[a]_R = [b]_R$.

Conversely, suppose that $[a]_R = [b]_R$. We will show that $aRb$. Since $R$ is reflexive, we know that $bRb$. So by definition of $[b]_R$, we conclude that $b \in [b]_R$. Since $[a]_R = [b]_R$ by assumption, we then have that $b \in [a]_R$. So by defintion of $[a]_R$, we conclude that $aRb$. $\square$

**Theorem.** *Let $R$ be an equivalence relation on $X$. Define $\mathcal{P}$ to be $\mathcal{P} = \{[x]_R : x \in X\}$. The collection $\mathcal{P}$ is then a partition of $X$.*

*Proof.* To prove that $\mathcal{P}$ is a partition of $X$, we have to show that

1. $[x]_R \neq \emptyset$ for all $x \in X$,
2. for any $a, b \in X$, if $[a]_R \cap [b]_R \neq \emptyset$ then $[a]_R = [b]_R$, and
3. $\bigcup \mathcal{P} = X$.

We prove item 1 as follows. Let $x$ be an arbitrary element of $X$. By the reflexivity of $R$, we conclude $xRx$. This means that the equivalence class $[x]_R$ is not empty since it contains at least an element, specifically element $x$.

We prove item 2 as follows. Suppose $a, b$ are any elements of $X$ such that $[a]_R \cap [b]_R \neq \emptyset$. Let $c \in ([a]_R \cap [b]_R)$. Then $c \in [a]_R$ and $c \in [b]_R$. Since $c \in [a]_R$, we see that $aRc$ by definition of $[a]_R$. Therefore, $[a]_R = [c]_R$ by last theorem. Since $c \in [b]_R$, we see that $bRc$ by definition of $[b]_R$. Therefore, $[b]_R = [c]_R$ by last theorem. (Therefore, $[c]_R = [b]_R$ as well since equality of sets is an equivalence relation.) Therefore, $[a]_R = [c]_R$ and $[c]_R = [b]_R$. So $[a]_R = [b]_R$ by transitivity of set equality.

We prove item 3 as follows. Since each element of $\mathcal{P}$ is a subset of $X$, we conclude that $\bigcup \mathcal{P} \subseteq X$. Now let $x$ be an arbitrary element of $X$. Then $x \in [x]_R \in \mathcal{P}$. Therefore, $x \in \bigcup \mathcal{P}$. Thus, $X \subseteq \bigcup \mathcal{P}$. Hence, $\bigcup \mathcal{P} = X$. $\square$

**Theorem.** *Let $\mathcal{C}$ be any partition of $X$. Define $R$ to be a relation on $X$ by declaring that for any $a, b \in X$, it holds that $aRb$ if and only if there exists a $C \in \mathcal{C}$ such that $a \in C$ and $b \in C$. The relation $R$ so defined is then an equivalence relation on $X$.*

*Proof.* ... $\square$

- For any relation $R$ from $A$ to $B$, we define its *inverse relation* $R^{-1}$ to be $\{(b, a) \subseteq B \times A : (a, b) \in R\}$.

- A relation $R$ on $X$ is called *antisymmetric* if for all $a, b \in X$ if $aRb$ and $bRa$, then $a = b$. A relation $R$ on $X$ is called *asymmetric* if for no $a \in X$ is $aRa$. A relation $R$ on $X$ is called a *partial order* if it is reflexive, transitive, and antisymmetric. A partial order $R$ on $X$ is a *total order* (or *linear order*) if for all $a, b \in X$, either $aRb$ or $bRa$. Let $R$ be a partial order on $X$. An element $a$ in $X$ is called a *minimal* element if for any element $b \in X$, if $bRa$ then $b = a$. Similarly, an element $a$ in $X$ is called a *maximal* element if for any element $b \in X$, if $aRb$ then $b = a$. An element $a$ in $X$ is called a *minimum* element if $aRb$ for all elements $b \in X$. An element $a$ in $X$ is called a *maximum* element if $bRa$ for all elements $b \in X$.

- **Exercises:**

  1. Let $S$ be a nonempty set, and let $\mathcal{C}$ be the collection of all subsets of $S$, i.e., $\mathcal{C} = \{C : C \subseteq S\}$. Prove that $\subseteq$ is a partial order on $\mathcal{C}$.

     Show that if $|S| > 1$, then $\subseteq$ is not a a total order.

  2. Let $X = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define $R$ to be a relation on $X$ by declaring that for all $a, b \in X$, we have $aRb$ if and only if $a$ is factor of $b$. For example, $2R8$, $3R9$, but it's not true that $9R8$. Prove that $R$ is a partial order on $X$.

     Show that $R$ is not a total order.

  3. Prove that $\leq$ is a total order on $\mathbb{R}$.

  4. For each of the partial orders listed above, list all elements that are minimal, maximal, minimum, or maximum.

# Functions

- A function $f$ from set $A$ to set $B$, written $f : A \to B$, is a relation from $A$ to $B$ such that each $a \in A$ is the first component of exactly one ordered pair. Set $A$ is called the *domain* and set $B$ is called the *codomain* of $f$. If $(a, b) \in f$, we write $b = f(a)$ and call $b$ the *image of $a$ under $f$*.

  The *range of $f$* is defined to be $\{b \in B : b = f(a) \text{ for some } a\}$.

- A function $f$ is *injective (1-to-1)* if $f(a) = f(a') \implies a = a'$ for all $a, a'$.

  A function is *surjective (onto)* if its range equals its codomain.

  A function is *bijective (1-to-1 and onto)* if it is both injective and surjective.

- **Theorem.** A function $f : A \to A$ that is 1-1 (onto) is not necessarily onto (1-1) unless $A$ is finite.

  *Proof.* ... □

- Given functions $f : A \to B$ and $g : B \to C$ we define the *composite function $g \circ f$* to be the function from $A$ to $C$ such that $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

- **Theorem 2.4** (Appendix 2, CZ). If $f : A \to B$ and $g : B \to C$ are bijective, then $g \circ f$ is bijective.

  *Proof.* ... □

- **Theorem 2.5** (Appendix 2, CZ). If $f : A \to B$ is a function, then $f^{-1}$ is a bijective function if and only if $f$ is bijective.

  *Proof.* ... □

- A permutation of $A$ is any bijective $f : A \to A$.

- $|A| = |B|$ if there is a bijection $f : A \to B$

# Sequences

- Let $S$ be any nonempty set. A infinite sequence in $S$ is a function from $\mathbb{N}$ to $S$. A finite sequence in $S$ is a function from $\{0, 1, 2, \ldots, n\}$ to $S$ for some $n \in \mathbb{N}$.

- We usually write a sequence by enumerating its range like so

  $$\langle a_0, a_1, a_2, \ldots \rangle$$

  or like so

  $$(s_0, s_1, s_2, s_3, s_4, s_5)$$

  or even like so

  $$v_0, v_1, v_2, v_3, v_4.$$

- We sometimes start counting from 1 instead of 0. So it's common to also see a sequence written as

  $$v_1, v_2, v_3, v_4, v_5$$

  etc.