MCS-236: Graph Theory Handout #Ch6
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Nov 3, 2010

Chapter 6: Traversability

6.1 Eulerian Graphs

Definitions. An *Eulerian circuit* of a graph G is a closed trail containing every edge of G. An *Eulerian graph* is a connected graph containing an Eulerian circuit. An *Eulerian trail* of a graph G is an open trail containing every edge of G.

Theorem (Theorem 6.1 of CZ). *A nontrivial, connected graph G is Eulerian if and only if every vertex of G has even degree.*

Proof. Only if: ...
If: Assume graph G is nontrivial, connected, and every vertex of it has even degree. We execute the following algorithm on G.

```
H ← G
T ← ⟨s⟩, where s is some vertex of H
while H is not an empty graph do {
   x ← any vertex on T incident with some edge of H
   T′ ← maximal trail in H starting at x (thus ending at x as well)
   insert T′ into T at x
   H ← H − E(T′) }
return T as the desired circuit
```

It can be shown that this algorithm works correctly by proving three points.

1. The algorithm never gets stuck.
2. The algorithm terminates.
3. When the algorithm terminates, T is an Eulerian circuit of G.

□

Corollary (Corollary 6.2 of CZ). *A connected graph G contains an Eulerian trail if and only if exactly 2 vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.*

Proof. . . . □

Example 6.3. Find a necessary and sufficient condition for the graph $G \times H$ of nontrivial connected graphs G and H to be Eulerian.
Answer. Either both G and H are Eulerian, or every vertex of G and H is odd.

Note. The theorem and corollary of this section are correct for multigraphs as well.

6.2 Hamiltonian Graphs

Definitions. A *Hamiltonian cycle* of a graph G is a spanning cycle of G, i.e., a cycle that contains every vertex. A *Hamiltonian graph* is a graph that contains a Hamiltonian cycle. A *Hamiltonian path* in a graph G is a spanning path of G, i.e., a path that contains every vertex.

Theorem (Theorem 6.4 of CZ). *The Petersen graph is non-Hamiltonian.*

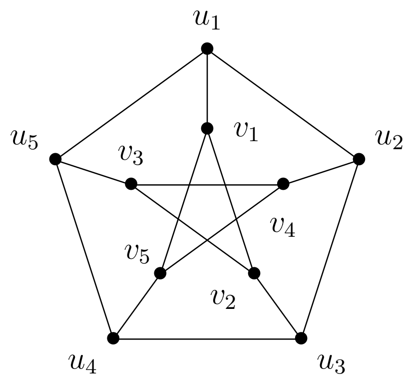
Proof. . . . □

Note. Lines 7–8 in the proof of Theorem 6.4 of CZ on page 143 states, "Without loss of generality, assume that C' contains at least three edges of C." We will validate this claim. The justification for this statement is as follows: If C' doesn't contain at least three edges of C, but it fact C'' does, then it is possible to redraw any labeled Petersen Graph in such a way that C' and C'' are switched, i.e., the inner cycle becomes an outer one, and vice versa. We then apply the same reasoning that follows to the redrawn graph.

Now to the switching. Consider the Petersen Graph $PG = (V, E)$ with vertex set

$$V = \{\, u_i, v_i : 1 \leq i \leq 5 \,\}$$

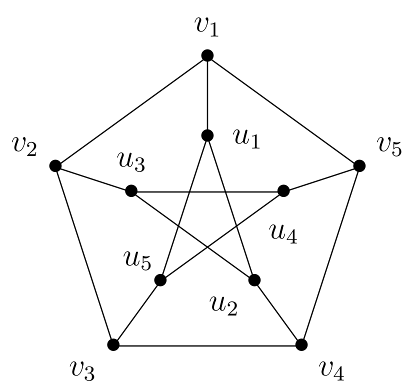
and edges as depicted in this diagram.



Let $\varphi : V \to V$ be defined by

$$\begin{array}{lllll} \varphi(u_1) = v_1 & \varphi(u_2) = v_5 & \varphi(u_3) = v_4 & \varphi(u_4) = v_3 & \varphi(u_5) = v_2 \\ \varphi(v_1) = u_1 & \varphi(v_2) = u_2 & \varphi(v_3) = u_3 & \varphi(v_4) = u_4 & \varphi(v_5) = u_5 \end{array}$$

It's straightforward to check that φ is an automorphism (an isomorphism from the vertex set of a graph onto inself). In fact, more can be said about φ. This automorphism φ maps the "inner cycle" into the "outer cycle" and vice versa! The following diagram depicts $\varphi(PG)$, drawn in such way as to reflect this fact. (In the new drawing, each vertex x in the previous drawing is labeled by its image $\varphi(x)$.)



This is why it is immaterial which one of the "two cycles" of PG is considered "outer" and which one "inner."

Theorem (Theorem 6.5 of CZ)**.** *Let G be a Hamiltonian graph and let S be such that $\emptyset \subset S \subset V(G)$. Let $G - S$ have k connected components. Then $k \leq |S|$.*

Proof. Assume the hypotheses and suppose for the sake of contradiction that $k > |S|$. Since G is Hamiltonian, it contains some spanning cycle; let C be a spanning cycle.

Let the graph $C - S$ consists of ℓ connected components, say $P_1, P_2, \ldots, P_\ell$, where each component is a path. We see that $|S| \geq \ell$.
For each component i of $G - S$, where $1 \leq i \leq k$, choose a representative vertex v_i. Since $k > |S| \geq \ell$, the Pigeonhole Principle asserts that some component P_r of $C - S$ contains at least two distinct vertices v_s and v_t. In $C - S$, vertex v_s is connected to v_t since path P_r passes through them. Since $C - S$ is a subgraph of $G - S$, path P_r is a path in $G - S$ as well. Thus, vertices v_s and v_t belong to the same component of $G - S$, contradicting the fact that they are representatives of distinct components of $G - S$. □

Corollary. *No Hamiltonian graph has a cut vertex.*

Proof. ... □

Theorem (Bondy & Chvatal, Theorem 6.8 of CZ)**.** *Let u and v be nonadjacent vertices in a graph G of order n such that* $\deg u + \deg v \geq n$. *Then $G + uv$ is Hamiltonian if and only if G is.*

Proof. Suppose G is Hamiltonian. Then there exists a spanning cycle C in G. Cycle C is a spanning cycle in $G + uv$ as well. Hence, $G + uv$ is Hamiltonian.
Conversely, suppose $G + uv$ is Hamiltonian and let C be a spanning cycle in it. If C doesn't contain edge uv, then C is a spanning cycle in G as well, and thus G is Hamiltonian. So assume from now on that C contains edge uv. Deleting edge uv from C gives a $u - v$ path P that spans $G + uv$. Path P spans G as well. Say that P is the path $u = v_1, v_2, \ldots, v_n = v$. We know $n \geq 3$ since $G + uv$ is Hamiltonian so it contains a spanning cycle; and every cycle has length at least 3. We observe that $n > 3$ because if $n = 3$, then $\deg_G v_1 + \deg_G v_3 = 2 < 3 = n$ contradicting $\deg_G u + \deg_G v \geq n$. So, in particular, we know $v_2 \neq v_{n-1}$.

Claim. There exists an index j, where $3 \leq j \leq n - 1$, such that $v_1 v_j$ and $v_{j-1} v_n$ are edges of G.

Proof of claim. Suppose not. Let k be the number of neighbors of v_1 in $\{v_3, v_4, \ldots, v_{n-1}\}$. Let ℓ be the number of neighbors of v_n in $\{v_2, v_3, \ldots, v_{n-2}\}$. Since the claim doesn't hold by supposition, exactly k vertices in $\{v_2, v_3, \ldots, v_{n-2}\}$ are *not* allowed to be neighbors of v_n. Thus

$$\ell \leq |\{v_2, v_3, \ldots, v_{n-2}\}| - k = (n - 3) - k,$$

i.e.,

$$k + \ell \leq n - 3.$$

Since $v_1 v_2 \in E(G)$ and $v_1 v_n \notin E(G)$, we see that

$$\deg_G u = \deg_G v_1 = k + 1.$$

Since $v_{n-1} v_n \in E(G)$ and $v_1 v_n \notin E(G)$, we see that

$$\deg_G v = \deg_G v_n = \ell + 1.$$

Thus, we have

$$(k + 1) + (\ell + 1) \leq (n - 3) + 2$$

i.e.,

$$\deg_G u + \deg_G v \leq n - 1$$

contradicting the assumption that $\deg_G u + \deg_G v \geq n$. □

Let j be an index that exists by the claim. Then $v_1, v_j, v_{j+1}, \ldots, v_n, v_{j-1}, v_{j-2}, \ldots, v_1$ is a spanning cycle of G. Thus, G is Hamiltonian. □

Definition. The closure $C(G)$ of a graph G is the graph obtained by, starting from G, repeatedly joining any pair of nonadjacent vertices whose degree sum in the current graph is at least n, until no such pair exists.

Theorem (Theorem 6.9 of CZ)**.** *Graph G is Hamiltonian if and only if $C(G)$ is.*

Proof. ... □

Corollary (Corollary 6.10 of CZ)**.** *If a graph G has $n \geq 3$ and $C(G)$ is complete, then G is Hamiltonian.*

Proof. ... □

Theorem (Ore, Theorem 6.6 of CZ)**.** *Let G be a graph of order $n \geq 3$. If $\deg u + \deg v \geq n$ for each pair of nonadjacent vertices of G, then G is Hamiltonian.*

Proof. Since $C(G)$ is complete, the result follows by the preceding Corollary. □

Theorem (Dirac, Theorem 6.7 of CZ)**.** *Let G be a graph of order $n \geq 3$. If $\deg v \geq n/2$ for each vertex v of G, then G is Hamiltonian.*

Proof. By Ore's Theorem. □