# Chapter 10: Coloring

# Four Color Theorem (4CT)

The Four Color Problem appeared in writing for the first time in 1852. It asks whether it is possible to color any map using 4 colors if any two countries sharing a border must receive different colors. Many have attempted and failed to solve the problem but in the process advanced graph theory. The problem was finally settled in the affirmative in 1976 with the help of a computer. Even today all proofs of 4CT involves case-by-case checking by a computer.

# Vertex Coloring

**Definitions.**

A *proper coloring* of a graph is an assignment of colors to the vertices, one color per vertex, in such a way that no adjacent vertices receive the same color. A proper coloring that uses no more than $k$ colors is called a *k-coloring*. A *color class* is the set of all vertices that are assigned the same color. A graph is *k-colorable* if it admits a $k$-coloring. The *chromatic number* $\chi(G)$ of a graph $G$ is the least $k$ such that $G$ is $k$-colorable. A graph $G$ is *k-chromatic* if $\chi(G) = k$. A *minimum coloring* of a graph $G$ is a $\chi(G)$-coloring of it.

A subset $U$ of $V(G)$ is an *independent set* if no two vertices of $U$ are adjacent. A *maximum independent set* of $G$ is an independent set of maximum size. The *vertex independence number* $\beta(G)$ of a graph $G$ is the size of a maximum independent set of $G$. A *clique* is a complete subgraph of $G$. The *clique number* $\omega(G)$ of a graph $G$ is the order of a largest clique in $G$.

Note that an equivalent way to define proper coloring is to say that it is a partition of the vertex set into independent subsets.

**Observations.**

1. We may restrict our study of coloring problems to connected, simple graphs.

2. If $G$ is a subgraph of $H$, then $\chi(G) \le \chi(H)$.

3. $1 \le \chi(G) \le n$ for any graph $G$.

To show that $\chi(G) = k$,

1. show $\chi(G) \le k$ by exhibiting a $k$-coloring, and

2. show $\chi(G) \ge k$ by proving that $G$ is not $(k-1)$-colorable.

**Exercise.** Show that

1. $\chi(C_n) = 2$ if $n$ is even, and $\chi(C_n) = 3$ if $n$ is odd.

2. For a graph $G$ of order $n$, $\chi(G) = n$ iff $G \cong K_n$.

3. $\chi(W_{1,k}) = 3$ if $k$ is even, and $\chi(W_{1,k}) = 4$ if $k$ is odd.

**Theorem** (Theorem 10.2 of CZ). $\chi(G) = 2$ *if and only if $G$ is a nonempty bipartite graph.*

*Proof.* (Sketch) $\Rightarrow$: The 2 color clases in the $\chi$-coloring gives a bipartition; $\chi > 1$ implies $G$ is not empty.
$\Leftarrow$: Bipartiteness implies $\chi \le 2$. Nonemptiness implies $\chi \ge 2$. $\qquad\square$

**Theorem** (Theorem 10.5 of CZ). $\chi \ge \omega$ *and* $\chi \ge n/\beta$.

*Proof.* (Sketch) $\chi \ge \omega$ by existence of a clique of size $\omega$. $\chi \ge n/\beta$ since each color class is an independent set, and each independent set has size $\le \beta$. $\qquad\square$

**Theorem** (Theorem 10.7 of CZ). $\chi \le \Delta + 1$.

*Proof.* (Sketch) By greedy coloring algorithm. $\qquad\square$

**Theorem** (Brooks, Theorem 10.8 of CZ). *A connected graph $G$ that's not an odd cycle or a complete graph satisfies $\chi \le \Delta$.*

**Theorem** (Theorem 10.9 of CZ). *For every graph $G$,*

$$\chi(G) \leq 1 + \max\{\,\delta(H) : H \text{ is an induced subgraph of } G\,\}$$

*Proof.* (Sketch) By reverse minimum-degree vertex coloring algorithm. □

**Theorem** (Theorem 10.10 of CZ). *For every integer $k \geq 3$, there exists a triangle-free graph with chromatic number $k$.*

*Proof.* (Sketch) By constructing the Mycielski graphs. □

**Note.** Map coloring problem reduces to the vertex coloring problem via the *dual* of a plane graph. The 4CT when stated as a vertex coloring problem asserts that every planar graph is 4-colorable.

# Edge Coloring

**Definitions.**
The definitions of *proper edge coloring*, *k-edge-coloring*, *color class*, *k-edge-colorable*, *edge chromatic number* (or *chromatic index*) $\chi_1(G)$, *k-edge-chromatic*, and *minimum edge coloring*, etc., parallel those of the corresponding vertex versions.

**Observations.**

1. We may restrict our study of edge coloring problems to connected graphs.

2. Edge coloring results of multigraphs are usually different from those of simple graphs.

3. If $G$ is a subgraph of $H$, then $\chi_1(G) \leq \chi_1(H)$.

4. $\chi_1(G) \geq \Delta(G)$ for any graph $G$.

**Theorem** (Vizing, Theorem 10.12 of CZ). *For simple graphs, $\chi_1 = \Delta$ or $\Delta + 1$.*

**Definition** A *matching* of a graph $G$ is a subset of the edges of $G$, no two of which are adjacent.

**Exercise.** Show that $\chi_1(C_n) = 2$ if $n$ is even, and $\chi_1(C_n) = 3$ if $n$ is odd.

**Theorem** (Theorem 10.13 of CZ). $m > \lfloor n/2 \rfloor \Delta$ *implies* $\chi_1 = \Delta + 1$.

*Proof.* (Sketch) An edge coloring partitions the edge set into a collection of matchings, so no color class has more than $\lfloor n/2 \rfloor$ edges. $\qquad\square$

**Theorem** (Theorem 10.15 of CZ). *For* $n \geq 2$, $\chi_1(K_n) = n$ *if* $n$ *is odd, and* $\chi_1(K_n) = n-1$ *if* $n$ *is even.*

*Proof.* (Sketch) By Theorem 10.13 and Berge's coloring. $\qquad\square$

**Theorem** (Konig's, Theorem 10.17 of CZ). *If* $G$ *is a nonempty bipartite graph, then* $\chi_1(G) = \Delta(G)$.

*Proof.* (Sketch) For each edge $e = uv$, if there's a color $c$ that is missing at both $u$ and $v$, then assign color $c$ to $e$. Otherwise, say $u$ misses color 1 and $v$ misses color 2. Switch the colors of the $1, 2$-path starting at $v$; then assign color 1 to $e$. $\qquad\square$

**Note.** Edge coloring problem reduces to vertex coloring problem via the line graph.