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Characterization of Bipartite Graphs

CZ p.21–22 on bipartite graphs contains a bug. We will fix the bug and also offer an alternate proof.

Definition A nontrivial graph G is called *bipartite* if its vertex set can be partitioned into 2 nonempty subsets U and W, called *partite sets*, such that every edge of G joins a vertex of U and a vertex of W.

Theorem 1. *If a nontrivial, connected graph G is bipartite, then G contains no odd cycles.*

Proof. Suppose G is a nontrivial, connected, bipartite graph containing an odd cycle $v_1, v_2, \ldots, v_{2k+1}, v_1$, for some integer k. Let V_o and V_e be the partite sets. Wlog, say $v_1 \in V_o$. Since v_1v_2 is an edge, $v_2 \in V_e$. Since v_2v_3 is an edge, $v_3 \in V_o$. Continuing this reasoning $2k$ times we conclude that $v_{2k+1} \in V_o$. Since $v_{2k+1}v_1$ is an edge, $v_1 \in V_e$. Now we have $v_1 \in V_o$ and $v_1 \in V_e$, a contradiction. □

Lemma 1. *Every nontrivial tree T is bipartite.*

Proof. We will describe an algorithm for partitioning the vertices of T into partite sets U and W. Pick a vertex u of T and put it in U. While there exists some vertex that has not been put into U or W yet, pick such a vertex v that is adjacent to some vertex in U or in W. Such a vertex exists since T is connected. Note that v is adjacent to exactly one vertex in $U \cup W$, since otherwise we would have a cycle, contradicting T being a tree. If v is adjacent to a vertex in U, we put v in W; and if v is adjacent to a vertex in W, we put it in U. The algorithm must terminate since the graph is finite, and when it terminates we have a bipartition showing bipartiteness of T. □

Lemma 2. *If a nontrivial, connected graph G contains no odd cycles, then G is bipartite.*

Proof. Let T be a spanning tree of G. By Lemma 1 tree T is bipartite. We will show that the partite sets in the bipartition of T also proves G bipartite. Let's call an edge e

of G a *tree edge* if e is an edge of T, and call it a *non-tree edge* otherwise. Every tree edge joins vertices of different partite sets since the bipartition is for T. Let $e = uv$ be any non-tree edge. Then $T + e$ has exactly one cycle, containing e. Since every cycle of G is even, the $u - v$ path in T has an odd number of edges, thus an even number of vertices. Thus, u and v belong to different partite sets. This completes the proof. □

Theorem 2. *If a nontrivial graph G contains no odd cycles, then G is bipartite.*

Proof. There are 2 cases.

Case 1. Every component of G is trivial. We choose an arbitrary vertex u and put $U = \{u\}$, and let $W = V(G) \setminus U$. Then U, W is a bipartition proving G bipartite.

Case 2. Some component of G is nontrivial. Say G has k nontrivial components, for some $k \geq 1$. Let G_i $(1 \leq i \leq k)$ be the k nontrivial components. By Lemma 2, each component G_i is bipartite; say U_i and W_i are its partite sets. We put

$$U = \bigcup \{U_i : 1 \leq i \leq k\} \cup \{v \in V(G) : v \text{ is not in any } G_i, \text{ for all } 1 \leq i \leq k\}$$

and let

$$W = \bigcup \{W_i : 1 \leq i \leq k\}.$$

Then U, W is a bipartition proving G bipartite. □

Theorem 3 (CZ Theorem 1.12)**.** *A nontrivial graph G is bipartite if and only if G contains no odd cycles.*

Proof. By Theorem 1 and Theorem 2. □