San Skulrattanakulchai
Gustavus Adolphus College

Chapter 2

The proof of Theorem 2.4 in CZ has a gap, which we'll fill up here.

Theorem (CZ Theorem 2.4)**.** *Let G be a graph of order n. If $\deg u + \deg v \geq n - 1$ for every two nonadjacent vertices u and v of G, then G is connected and* $\operatorname{diam}(G) \leq 2$.

Proof. Let G be a graph satisfying the hypothesis. Choose any two vertices u, v. If u is adjacent to v, then $\langle u, v\rangle$ is a $u - v$ path of length 1. If u is not adjacent to v, then $|N(u) \cup N(v)| \leq n - 2$, which implies

$$-|N(u) \cup N(v)| \geq 2 - n.$$

By assumption

$$|N(u)| + |N(v)| \geq n - 1.$$

Therefore,

$$\begin{aligned} |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &\geq (n-1) + (2-n) \\ &= 1. \end{aligned}$$

This says that u and v have some common neighbor. Let w be a common neighbor. Then $\langle u, w, v\rangle$ is a $u - v$ path of length 2.

Therefore, between any two distinct vertices in G there's a path of length 1 or 2 connecting them. In other words, G is connected and $\operatorname{diam}(G) \leq 2$. □

Definition Let G be a graph. The degrees of its vertices listed (in any order) as a sequence of integers is called a *degree sequence* of G.

A sequence s of integers is called *graphical* if there exists some graph having s as its degree sequence.

Let G be a graph containing distinct vertices v, w, x, y, and containing edges vw and xy, but not containing edges wx or vy. Let $G' = G - \{vw, xy\} + wx + vy$. We say that G' is obtained from G by performing a *2-switch*. Note that graphs G and G' have exactly the same degree sequence(s).

Lemma. *Let $s : d_1, d_2, \ldots, d_n$ be a sequence of integers. The followings are true.*

(i) For any s' where s' is a permutation of s, the sequence s is graphical if and only if s' is.

(ii) If $d_i = 0$ for all $1 \leq i \leq n$, then s is graphical.

(iii) If $d_i < 0$ or $d_i \geq n$ for some $1 \leq i \leq n$, then s is not graphical.

Proof. (i) holds because the definition of degree sequence allows arbitrary ordering of the vertices when recording the degrees.

(ii) holds because s is the degree sequence of the empty graph of order n.

(iii) holds because every vertex v in a graph of order n satisfies $0 \leq \deg v \leq n - 1$.

□

The next theorem was proved in CZ by contradiction. We do it directly by algorithm.

Theorem (CZ Theorem 2.10)**.** *Let n be a positive integer ≥ 2. Let s be a non-increasing sequence*

$$d_1, d_2, \ldots, d_n$$

of nonnegative integers such that $d_1 < n$.. Let t be the sequence that results from s by removing d_1 and subtracting 1 from the next d_1 numbers in s. That is, t is the sequence

$$d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n.$$

Then s is graphical if and only if t is graphical.

Proof. First assume t is graphical. Let H be a graph having t as its degree sequence. Name the vertices of H using integers $2, 3, \ldots, n$ so that $\deg i = d_i$ for all $2 \leq i \leq n$. Let G be the graph obtained from H by adding vertex 1, and adding edges joining 1 to j for all $2 \leq j \leq d_1 + 1$ (for a total of d_1 edges). We see that G has s as its degree sequence. Thus, s is graphical.

Next assume s is graphical. Let G have s as its degree sequence. Let W be the set of d_1 consecutive integers $\{2, 3, \ldots, d_1, d_1 + 1\}$. Name the vertices of G using integers $1, 2, 3, \ldots, n$ so that (i) $\deg i = d_i$ for all $1 \leq i \leq n$, and (ii) $|N(1) \cap W|$ is maximized (over all possible ways of naming the vertices using $1, 2, 3, \ldots, n$). From G we will create a graph G^* such that (i) G^* has exactly the same degreee sequences as G, and (ii) $N(1) = W$.

If $N(1) = W$, let $G^* := G$. So assume from now on that $N(1) \neq W$. Let $w \in W$ be any non-neighbor of 1, and let $x \notin W$ be any neighbor of 1. Since s is non-increasing, $d_w \geq d_x$. Now d_w cannot be equal to d_x because our choice of vertex naming maximizes $|N(1) \cap W|$. Therefore, $d_w > d_x$, which means there exists some vertex y that is a neighbor of w but not a neighbor of x. We now have vertices $1, w, x, y$ such that $1x$ and wy are edges of G but $1w$, xy are not edges. Performing a 2-switch on these four vertices results in a graph G' with the same degree sequences as G. However, the quantity $|N(1) \cap W|$ in G' is one more than the same quantity in G. If now $N(1) = W$ in G', we let $G^* := G'$. If not, we repeat this same reasoning and operation to change G' to another graph having $|N(1) \cap W|$ one more than G' has. We have to repeat this process no more than d_1 times until $N(1) = W$, and we call this last graph obtained this way G^*. Let $H = G^* - 1$. Then H has t as its degree sequence; thus, t is graphical. $\square$