MCS-236: Graph Theory Handout #Ch5
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Connectivity

5.1 Cut Vertices

Definition 1. A vertex v in a connected graph G is a *cut vertex* if $G - v$ is disconnected.
Definition 2. A vertex v in a graph G is a *cut vertex* if $G - v$ has more connected components than G.

Exercise. Prove that these two definitions are equivalent.

Lemma (End-Vertex Lemma). *If v is an end vertex in a graph G, then v is not a cut vertex of G.*

Proof. Let v belong to the connected component H of G. Grow a maximal path P starting from v. Since $\deg v = 1$, vertex v is one of the two ends of P. By the Maximal Path Theorem, $H - v$ is connected. Thus v is not a cut vertex. □

Theorem (CZ, Theorem 5.1). *Let G be a graph containing a bridge e incident with vertex v. Vertex v is a cut vertex if and only if $\deg v \geq 2$.*

Proof. Let G be a graph containing a bridge e incident with vertex v.
$\Rightarrow$: Suppose $\deg v < 2$. Since e is incident with v, we have $\deg v \geq 1$. Thus, $\deg v = 1$. By the End-Vertex Lemma, v is not a cut vertex.
$\Leftarrow$: Suppose $\deg v \geq 2$ but v is not a cut vertex. Let bridge e join v to w and let H be the connected component of G that contains v. Since $\deg v \geq 2$, vertex v is adjacent to some vertex x that is different from v or w. Therefore, in H, vertex w is connected to vertex x via the path $P : w, v, x$. This says that vertices v, w, and x are all in the same component H. By assumption, vertex v is not a cut vertex of G; thus v is not a cut vertex of H, i.e., $H - v$ is connected. This says that all vertices in $H - v$ are in the same component. Hence, vertex w is connected in $H - v$ to vertex x via some path Q. Concatenating P to Q gives a cycle in H. This cycle contains edge e, contradicting the fact that e is a bridge. □

Corollary (CZ, Corollary 5.2)**.** *Let G be a connected graph of order 3 or more. If G contains a bridge, then G contains a cut-vertex.*

Proof. Let G be a connected graph of order at least 3 and let $e = vw$ be a bridge in G. By the Bridge Lemma, $G - e$ consists of two components: component G_v containing v and component G_w containing w. Since G has at least 3 vertices, at least one of G_v and G_w has more than one vertex. Assume wlog that G_v has more than one vertex. Thus, $\deg_{G_v} v \geq 1$. Since v is adjacent in G to w but w is not in G_v, we conclude that $\deg_G v \geq 2$. By Theorem 5.1 v is a cut vertex (in G). □

Theorem (CZ, Corollary 5.4)**.** *A vertex v of a connected graph G is a cut vertex of G if and only if there exist vertices u and w distinct from v such that v lies on every $u - w$ path in G.*

Proof. Suppose v is a cut vertex of a connected graph G. Then $G - v$ is disconnected, i.e., it has at least 2 connected components. Let u be any vertex in $G - v$ and let w be any other vertex in $G - v$ belonging to some component different from u's. Since G is connected, there exist some $u - w$ path in G. However, there exist no $u - w$ path in $G - v$ because u and w come from different components of $G - v$. This implies that each $u - w$ path in G passes through v, since $G - v$ differs from G only in that $G - v$ misses vertex v and all edges incident to v.

Conversely, suppose v is any vertex in a connected graph G and G contains some vertices u, w such that v lies on every $u - w$ path in G. This assumption implies that any $u - w$ path in G can no longer be a $u - w$ path when vertex v and all its incident edges are deleted from G. Thus, $G - v$ is disconnected since it has vertices u and w that are not connected. Hence, v is a cut vertex of G. □

Theorem (CZ, Corollary 5.6)**.** *Every nontrivial connected graph contains at least two vertices that are not cut vertices.*

Proof. This is just Theorem 1.9 (with the phrase "connected graph of order 3 or more" changed to "connected nontrivial graph") restated in terms of cut vertices. See Handout #5. □

5.2 Blocks

Definition. A *nonseparable graph* is nontrivial, connected, and has no cut vertex.

Note: K_2 is the only nonseparable graph of order less than 3.

Theorem (CZ, Theorem 5.7)**.** *A graph of order at least 3 is nonseparable if and only if every two vertices lie on a common cycle.*

Proof. Let G be a graph of order at least 3.
Suppose every two vertices of G lie on a common cycle. Graph G is nontrivial since its order is at least 3. Let x, y be two vertices of G. By assumption there is a cycle C that contains both x and y. Thus there is an $x-y$ path along C. Hence G is connected. Fix a vertex v. Let u, w be any two vertices distinct from v. By assumption there is a cycle that contains both u and w. This cycle gives two internally disjoint $u-w$ paths, at least one of which does not go through v. Hence, v is not a cut vertex by Corollary 5.4. Thus, graph G contains no cut vertex. Therefore, G is nonseparable.
Conversely, suppose G is nonseparable. Let u be a vertex of G. We will prove that if v is any vertex of G distinct from u, then there is a cycle that goes through both u and v, by induction on the distance $d(u, v)$. First suppose that $d(u, v) = 1$, i.e., G has an edge e joining u to v. Since G has order at least 3 and it contains no cut vertex (because G is nonseparable), we conclude by Corollary 5.2 that G contains no bridge. Therefore, e is not a bridge; and thus some cycle C contains e. Hence, cycle C contains both u and v (since e joins u to v). Next suppose that $d(u, v) = k > 1$ and assume inductively that for any vertex x, where $0 < d(u, x) < k$, there exists some cycle that goes through both u and x. Let $P : u = v_0, v_1, \ldots, v_{k-1}, v_k = v$ be a $u-v$ geodesic. Since $0 < d(u, v_{k-1}) = k-1 < k$, there exists, by inductive assumption, a cycle C that passes through both u and v_{k-1}. If cycle C goes through v_k as well, then we are done. So assume from now on that C does not go through v_k. Since G contains no cut vertex, v_{k-1} is not a cut vertex. This means that there exists some path in $G - v_{k-1}$ (and in G as well) connecting v_k to some vertex $x \in V(C) \setminus v_{k-1}$. Let Q be such a path of shortest possible length. Appending the $x - v_{k-1}$ path in C that goes through u (this path is unique if $x \neq u$) to Q gives a $v - v_{k-1}$ path P' in G that goes through u. Path P' together with edge $v_{k-1}v_k$ gives a cycle in G that goes through both u and v. Our claim follows by induction. □

Definition. Let G be a graph of positive size. Define a relation R on $E(G)$ as follows. For any edges e and f, declare eRf iff $e = f$ or there is a cycle in G that contains both e and f.

Theorem (CZ, Theorem 5.8)**.** *The relation R is an equivalence relation.*

Proof. ... □

Definition 1. A *block* of G is a nonseparable subgraph of G that is not a proper subgraph of any other nonseparable subgraph of G.
Definition 2. A *block* of G is a subgraph of G induced by the edges in an equivalence class defined by the relation R defined above.

Theorem (CZ, Exercise 5.15)**.** *The two definitions of block are equivalent.*

Proof. ... □

Corollary (CZ, Corollary 5.9)**.** *Every two distinct blocks B_1 and B_2 in a nontrivial connected graph G have the following properties:*
(a) The blocks B_1 and B_2 are edge-disjoint.
(b) The blocks B_1 and B_2 have at most one vertex in common.
(c) If B_1 and B_2 have a vertex v in common, then v is a cut vertex of G.

Proof. (a) By definition 2 of block, the edges $E(B_1)$ of block B_1 and the edges $E(B_2)$ of block B_2 belong to different equivalence classes. Therefore, $E(B_1) \cap E(B_2) = \emptyset$.

(b) Assume for the sake of contradiction that $|V(B_1) \cap V(B_2)| \geq 2$. Since B_1 is connected (because it's a block), for any two vertices shared by the two blocks there exists a path in B_1 connecting them. Let P_1 be a shortest path in B_1 connecting any two shared vertices; say that P_1 connects v to w. Path P_1 is nontrivial since $v \neq w$. Since B_2 is connected (because it's a block), there exists a path P_2 in B_2 connecting v to w. Path P_2 is nontrivial since $v \neq w$. Concatenating P_1 to P_2 gives a cycle containing some edge e_1 in B_1 and some edge e_2 in B_2. Thus, e_1 and e_2 belong to the same block by definition 2 of block. This contradicts part (a).

(c) Let $v \in V(B_1) \cap V(B_2)$. Being a block, B_1 is connected and nontrivial; thus, there exists $u_1 \in V(B_1)$ adjacent to v. Being a block, B_2 is connected and nontrivial; thus, there exists $u_2 \in V(B_2)$ adjacent to v.

We'll show that every $u_1 - u_2$ path in G contains v. Assume for the sake of contradiction that there is some $u_1 - u_2$ path P in G not containing v. Path P together with v and edges u_1v and vu_2 give a cycle C containing both u_1v and vu_2. Hence, edges u_1v and vu_2 belong to the same equivalence class, i.e., same block. This contradicts part (a). Therefore, v is a cut vertex of G. □

5.3 Connectivity

Definitions. Let $G = (V, E)$ be a connected graph. A subset U of V is called a *vertex cut* if $G - U$ is disconnected. A *minimum vertex cut* of G is a vertex cut of least cardinality. The *(vertex) connectivity* $\kappa(G)$ of G is defined as follows: for a disconnected graph G, $\kappa(G) = 0$; for a connected graph G, $\kappa(G)$ equals the cardinality of a smallest vertex subset U such that $G - U$ is either disconnected or trivial. A graph G is *k-connected* if $\kappa(G) \geq k$. Note that any graph G satisfies $0 \leq \kappa(G) \leq n - 1$. Note also that for a connected graph G of order n, $\kappa(G) = n - 1$ if and only if $G \cong K_n$.

Let $G = (V, E)$ be a connected graph. A subset X of E is an *edge cut* if $G - X$ is disconnected. An edge cut X is *minimal* if no proper subset of X is an edge cut. A *minimum edge cut* is an edge cut of minimum size.

Note that a minimal edge cut is not necessarily minimum, but every minimum edge cut is necessarily minimal. (Prove!)

The following lemma characterizes minimal edge cuts.

Lemma (Minimal Edge Cut Lemma)**.** *If X is a minimal edge cut of a connected graph G, then $G - X$ contains exactly 2 components. Moreover, X consists of all the edges of G that join a vertex in one component to a vertex in another component.*

Proof. Assume G is a connected graph and X is a minimal edge cut of G. Choose any edge $e \in X$, say e joins u to v. We have $G - X$ is disconnected but $G - (X \setminus \{e\})$ is connected. This means that e is a bridge in $G - (X \setminus \{e\})$. By the Bridge Lemma, $G - (X \setminus \{e\}) - e$ consists of exactly two components, one containing u and the other containing v. Since $G - (X \setminus \{e\}) - e = G - X$, we conclude that $G - X$ consists of exactly two components, say G_1 and G_2, and every edge of X joins a vertex in G_1 to a vertex in G_2.

Let e be any edge in G joining a vertex in G_1 to a vertex in G_2. We'll show that $e \in X$. Suppose not. Then $G - X$ contains edge e. This contradicts G_1 and G_2 being distinct components in $G - X$. □

The *edge connectivity* $\lambda(G)$ of G is defined as follows: for a disconnected graph G, $\lambda(G) = 0$; for a connected graph G, $\lambda(G)$ equals the cardinality of a smallest edge subset X such that $G-X$ is either disconnected or trivial. A graph G is *k-edge-connected* if $\lambda(G) \geq k$. Note that any graph G satisfies $0 \leq \lambda(G) \leq n-1$.

CZ, Example 5.10 Show that $\lambda(K_n) = n - 1$.

Proof. ... □

Theorem (CZ, Theorem 5.11)**.** *For any graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof. ... □