San Skulrattanakulchai
Gustavus Adolphus College Nov 29, 2012

Chapter 7. Digraphs

Strong Digraphs

Definitions. A digraph is an ordered pair (V, E), where V is the set of vertices and E is the set of *arcs* or *directed edge*. Each arc (u, v) is an ordered pair of vertices u and v. We usually write uv for arc (u, v). For any vertices u and v, when both uv and vu are arcs we call them *antiparallel* arcs. Arc uv *leaves* u and *enters* v. An arc is *incident to* the vertex it leaves and the vertex it enters. The *outdegree*, od v, of vertex v is the number of arcs that leave v. The *indegree*, id v, of vertex v is the number of arcs that enter v.
Let U and W be disjoint subsets of V. An arc uw, where $u \in U$ and $w \in W$, is said to *leave* U and *enter* W.
An *oriented graph* D is a digraph without antiparallel edges. In other words, it is derived from an undirected graph G by giving a direction to every edge of G. In this case, we say that D is an *orientation* of G. A *directed version* of an undirected graph G is a digraph that results from G by replacing every edge of G by a pair of *antiparallel* arcs joining the two ends previously joined by the replaced edge. An *underlying graph* of a digraph D is the graph that results from replacing every arc or pair of antiparallel arcs by an undirected edge. A digraph $D_1 = (V_1, E_1)$ is a *subdigraph* of a digraph $D_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$.
A *symmetric digraph* is a digraph such that if uv is an arc then vu is also an arc. Symmetric digraphs can be modeled by undirected graphs.

Theorem (The First Theorem of Digraph Theory, Theorem 7.1 of CZ)**.** *Any digraph of order n and size m satisfies $\sum_1^n(\text{od } v_i) = \sum_1^n(\text{id } v_i) = m$.*

Proof. Each edge contributes 1 to the outdegree sum, and 1 to the indegree sum. □

Definitions. The following concepts for digraphs: *walk, trail, path, length of a walk, closed walk, open walk, circuit, cycle*, *distance $d(u, v)$ from u to v*, *geodesic*, *Eulerian trail*, *Eulerian circuit*, *Eulerian digraph*, *Hamiltonian path*, *Hamiltonian cycle*, and *Hamiltonian digraph*, are defined in exactly same way as for undirected graphs. (To get the directed definitions, subsitute everywhere the term "arc" for "edge" in the undirected definitions.) Some digraph concepts are different from the undirected ones. For instance, there are at least three flavors of connectedness for digraphs:

- A digraph is *(weakly) connected* if its underlying graph is connected.
- A digraph is *semi-connected* if for any vertices u, v there's a $u - v$ path or a $v - u$ path.
- A digraph is *strongly connected* or *strong* if for any vertices u, v there's a $u - v$ path and a $v - u$ path. Or equivalently, if for any vertices u, v there's a $u - v$ path. Or equivalently, if there exists a vertex r such that for any vertex v there's a $r - v$ path and a $v - r$ path.

Theorem (Theorem 7.2 of CZ)**.** *If a digraph D contains a $u - v$ walk of length ℓ, then D contains a $u - v$ path of length at most ℓ.*

Proof. (Sketch) Keep cutting cycles from the given walk until none remains. □

Theorem (Theorem 7.3 of CZ)**.** *A digraph D is strong if and only if D contains a closed spanning walk*

Proof. $\Rightarrow$: Assume D strong. Label the vertices $v_1, v_2, \ldots, v_n$. Since D is strong, for each $1 \leq i < n$, there's a $v_i - v_{i+1}$ path; and there's also a $v_n - v_1$ path. Concatenating these n paths gives a closed spanning walk.
$\Leftarrow$: Let W be a closed spanning walk. Let u and v be arbitrary vertices. Since W is spanning, both u and v are on W and there's a subwalk of W from u to v. By Theorem 7.2 there's a $u - v$ path. Therefore, D is strong. □

Theorem (Theorem 7.4 of CZ)**.** *A nontrivial connected digraph D is Eulerian if and only if* $\text{od } v = \text{id } v$ *for every vertex v of D*

Proof. Similar to the proof for undirected graphs. □

Theorem (Robbins, Theorem 7.5 of CZ)**.** *A nontrivial connected graphs G has a strong orientation if and only if G contains no bridge, i.e., G is 2-edge-connected.*

Proof. (Sketch) Depth-first search G. Direct each tree edge downward and direct each back edge upward. □

Tournaments

Definitions. A *tournament* is an orientation of a complete graph, i.e., it is a digraph such that for any distinct vertices u, v, exactly one of uv and vu is an arc. A tournament is *transitive* if uw is an arc whenever both uv and vw are arcs.

Theorem (Theorem 7.6 of CZ)**.** *A tournament is transitive if and only if it has no cycles.*

Proof. $\Rightarrow$: Assume tournament T is transitive but contains a cycle $v_1, v_2, \ldots, v_k, v_1$. If $k = 2$, then T contains antiparallel arcs v_1v_2 and v_2v_1, contradicting T being a tournament. Thus, $k \geq 3$. Since v_1v_2 and v_2v_3 are arcs, transitivity implies v_1v_3 is an arc. Now that we know v_1v_3 is an arc, and since v_3v_4 is an arc, again transitivity implies v_1v_4 is an arc. We reason like this $k-2$ times to conclude that v_1v_k is an arc. However, v_kv_1 is also an arc of the digraph (since it's part of the cycle). Thus, T contains antiparallel arcs v_1v_k and v_kv_1, contradicting T being a tournament.
$\Leftarrow$: Assume tournament T contains no cycle. Let uv and vw be arcs of T. Thus, $u \neq w$ since T cannot contain antiparallel arcs. The tournament cannot have wu as arc because if it did it would contain the cycle $\langle u, v, w, u\rangle$. Since it's a tournament and $u \neq w$, either uw or wu is an arc. Thus, uw is an arc. Hence, T is transitive. $\square$

Theorem (Theorem 7.7 of CZ)**.** *If u is a vertex of maximum outdegree in a tournament, then $d(u, v) \leq 2$ for every vertex v.*

Proof. Let u be a vertex of maximum outdegree in a tournament T. Assume for the sake of contradiction that some vertex y satisfies $d(u, y) \geq 3$. Define $V_1 = \{\, x \in V(T) : d(u, x) = 1 \,\}$. Then od $u = |V_1|$, $u \notin V_1$, $y \neq u$, and $y \notin V_1$. No arc can leave $\{u\} \cup V_1$ and enter y since existence of any such arc would make $d(u, y) \leq 2$, contradicting $d(u, y) \geq 3$. Since T is a tournament, any two distinct vertices have exactly one arc joining them. Hence, for each $w \in \{u\} \cup V_1$, there's an arc yw. Thus, od $y \geq |\{u\}| + |V_1| = 1 + \text{od } u$, contradicting u having maximum outdegree. $\square$

Theorem (Theorem 7.8 of CZ)**.** *Every tournament contains a Hamiltonian path.*

Proof. Let T be a tournament. We prove by induction on the order n of T. Recall that Hamiltonian path means spanning path.
If $n = 1$, then T is trivial and it has a trivial path that's also spanning. If $n = 2$, then T is an oriented K_2, and it contains a path of length 1 that spans T.

Suppose $n \geq 3$ and assume inductively that any tournament of order $n-1$ contains a spanning path. Let x be a vertex of T. Then $T-x$ is a tournament of order $n-1$. By inductive assumption, $T-x$ contains a spanning path. Say $P: v_1, v_2, \ldots, v_{n-1}$ is a spanning path of $T-x$. Path P is a path in T as well. If xv_1 is an arc of T, then x, P is a spanning path in T and we are done. If $v_{n-1}x$ is an arc of T, then P, x is a spanning path in T and we are done. So assume from now on that $xv_1 \notin E(T)$ and $v_{n-1}x \notin E(T)$. Since T is a tournament, it follows that $v_1x \in E(T)$ and $xv_{n-1} \in E(T)$. Moreover, for each i, where $2 \leq i \leq n-2$, either $v_ix \in E(T)$ or $xv_i \in E(T)$, but not both. Now, arc v_1x enters x and arc xv_{n-1} leaves x. Therefore, there exists some j such that $1 \leq j \leq n-2$, $v_jx \in E(T)$, and $xv_{j+1} \in E(T)$. We now have $\langle v_1, v_2, \ldots, v_j, x, v_{j+1}, \ldots, v_{n-1}\rangle$ as a path that spans T. $\square$

Theorem (Theorem 7.9 of CZ)**.** *Every vertex in a nontrivial strong tournament belongs to some triangle.*

Proof. Let v be a vertex in a nontrivial, strong tournament T. Define

$$U = \{\, u \in V(T) : vu \in E(T) \,\}$$

and

$$W = \{\, w \in V(T) : wv \in E(T) \,\}.$$

Since T is a nontrivial strong tournament, some arc enters v and some arc leaves v. Thus, $U \neq \emptyset$ and $W \neq \emptyset$.

Clearly, $\{\{v\} \cup U \cup W\} \subseteq V(T)$ since $v \in V(T)$ and U and W are defined as subsets of $V(T)$. Let $x \in V(T)$. If $x = v$, then $x \in \{v\}$. Suppose $x \neq v$. Since T is a tournament, either xv is an arc and vx is an arc, but not both. Thus, $x \in U$ or $x \in W$. Hence, $V(T) \subseteq \{\{v\} \cup U \cup W\}$. Therefore, $V(T) = \{v\} \cup U \cup W$.

Moreover, $v \notin U \cup W$ by the definition of U and W. Also, $U \cap W = \emptyset$ because if there were any vertex x such that $x \in U \cap W$, then both vx and xv are arcs, contradicting T being a tournament.

Thus $\{\,\{v\}, U, W\,\}$ is a partition of $V(T)$.

Since T is strong, any vertex in U can reach v via some path. Such a path exists since $U \neq \emptyset$. Such a path must have at least one arc that leaves U because $v \notin U$. Any arc leaving U either enters v or enters W since $\{\{v\}, U, W\}$ is a partition of $V(T)$. Now, there can be no arc uv leaving U and entering v, because such an arc together with arc vu that is known to exist would give a pair of antiparallel arcs, contradicting T being a tournament. Therefore, some arc leaves U and enters W. Let uw be such an arc. We now have a triangle $\langle v, u, w, v \rangle$. $\square$

Definitions. A digraph D of order $n \geq 3$ is *vertex-pancyclic* if for every vertex v and for every $\ell = 3, 4, \ldots, n$, digraph D has a cycle of length ℓ containing v.

Theorem (Moon's Theorem)**.** *Every nontrivial strong tournament is vertex-pancyclic.*

Proof. Induction on cycle length ℓ, where the base case holds by Theorem 7.9. $\square$

Theorem (Theorem 7.10 of CZ)**.** *A nontrivial tournament T is Hamiltonian if and only if T is strong.*

Proof. (Sketch) $\Rightarrow$: A spanning cycle allows any vertex to reach any other vertex via an arc of the cycle.
$\Leftarrow$: By Moon's Theorem. $\square$

Theorem (Theorem 7.11 of CZ)**.** *If T is a strong tournament of order ≥ 4, then there exists a vertex v such that $T - v$ is a strong tournament.*

Proof. By Moon's Theorem. $\square$