MCS-236: Graph Theory Handout #Ch9
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Chapter 9. Planarity

Definitions. A *plane graph* G or a *planar embedding* of G is a drawing of G on the plane in such a way that no two edges meet, except at their common ends. Graphs that do admit such an embedding are called *planar*; ones that don't are called *nonplanar*.
A *girth* of a graph is the length of any smallest cycle if any. An acyclic graph has girth ∞. Thus, $3 \leq \text{girth } G \leq \infty$ for any (simple) graph G.

Examples of planar graphs are paths, cycles, trees, and the complete bipartite graphs $K_{2,k}$.

Consider a plane graph. A *region* is a maximal connected area that remains when the edges & vertices are removed from the plane. The *boundary* of a region is the vertices and edges touching the region.

Theorem (Jordan Curve Theorem)**.** *A simple closed curve partitions the plane into two regions: a bounded interior region and an unbounded exterior region.*

Lemma (Lemma A)**.** *Any bridge is the boundary of exactly one region. Deleting a bridge (and any resulting isolated vertex) from a plane graph does not change the number of regions. Any nonbridge edge is the boundary of exactly two regions. Deleting a nonbridge edge from a plane graph decreases the number of regions by one.*

Theorem (Euler Identity, Theorem 9.1 of CZ)**.** *If G is a connected plane graph of order n, size m, and r regions, then $n - m + r = 2$.*

Proof. We prove by induction on the number of cycles in G. If G has 0 cycle, then G is a tree since G is connected by assumption. Thus, $m = n - 1$ and $r = 1$. Therefore, $n - m + r = n - (n - 1) + 1 = 2$ and the result holds in the base case.
Now let G have k cycles, where $k > 0$, and assume inductively that any connected plane graph having fewer than k cycles satisfies the statement of the theorem. Let e be an edge belonging to some cycle of G. The plane graph $G - e$ has n vertices, $m - 1$ edges, and $r - 1$ regions. Moreover, $G - e$ is connected and has fewer than k cycles. Therefore, by the inductive hypothesis the result holds for $G - e$, i.e., $n - (m - 1) + (r - 1) = 2$. This implies that $n - m + r = 2$, so the result holds for G as well. □

Theorem (Generalization of Theorem 9.2 of CZ). *Let g be a fixed integer ≥ 3. If G is a planar graph of order n, size m, girth $\geq g$, and $n \geq (g+2)/2$, then $m \leq \frac{g(n-2)}{g-2}$.*

Proof. Note that for any planar graph G_1, there is a connected, planar graph G_2 that is a supergraph of G_1. Thus we may assume that G is connected.

First, assume G has $< g$ edges. Then G is acyclic since it has girth $\geq g$ and so it has too few edges to contain any cycle. Therefore, G is a tree since it's also connected. Hence, $m = n - 1$. Since $n \geq (g+2)/2$ by assumption, we have

$$g + 2 \leq 2n$$

i.e.,

$$gn - 2n - g + 2 \leq gn - 2g$$

i.e.,

$$(g-2)(n-1) \leq g(n-2)$$

i.e.,

$$m = n - 1 \leq \frac{g(n-2)}{g-2}$$

and the conclusion of the theorem holds.

Next, assume G has $\geq g$ edges. Fix an embedding of G on the plane. For each region i (where $1 \leq i \leq r$) of the plane graph G, let m_i be the number of edges on its boundary. Since G has at least g edges, has girth $\geq g$, and is connected, we see that $m_i \geq g$ for each i. Thus $\sum_{i=1}^{r} m_i \geq gr$. Also, $\sum_{i=1}^{r} m_i \leq 2m$ because, by Lemma A, each bridge contributes 1 to the sum and each nonbridge contributes 2 to the sum. Thus, $gr \leq 2m$; hence, $r \leq 2m/g$. Combining this last inequality with Euler Identity we have

$$2 = n - m + r \leq n - m + \frac{2m}{g}$$

i.e.,

$$2g \leq gn - (g-2)m$$

i.e.,

$$(g-2)m \leq gn - 2g$$

i.e.,

$$m \leq \frac{g(n-2)}{g-2}$$

as desired. □

Theorem (Theorem 9.2 of CZ)**.** *If G is a planar graph of order n, size m, and $n \geq 3$, then $m \leq 3n - 6$.*

Proof. Every graph has girth at least 3. Putting $g = 3$ in the generalized Theorem 9.2 of CZ gives the result. □

Theorem. *If G is a bipartite planar graph of order n, size m, and $n \geq 3$, then $m \leq 2n-4$.*

Proof. A bipartite graph has girth at least 4. Putting $g = 4$ in the generalized Theorem 9.2 of CZ gives the result. □

Theorem (Corollary 9.3 of CZ)**.** *Every planar graph contains a vertex of degree ≤ 5.*

Proof. Let G be a planar graph of order n and size m. If $n \leq 6$, then every vertex has degree ≤ 5 and we are done. So assume $n > 6$. By Theorem 9.2, $m \leq 3n - 6$. Thus,

$$\frac{m}{n} \leq 3 - \frac{6}{n}$$

i.e.

$$\frac{2m}{n} \leq 6 - \frac{12}{n}$$

i.e.

$$\frac{2m}{n} < 6$$

since $\frac{12}{n}$ is positive. The last inequality says that the average degree of G is < 6. Therefore, there exists at least a vertex whose degree does not exceed the average, i.e., some vertex v has $\deg v \leq \frac{2m}{n} < 6$, i.e., $\deg v \leq 5$. □

Theorem (Corollary 9.4 of CZ)**.** *K_5 is nonplanar.*

Proof. By Theorem 9.2. □

Theorem (Theorem 9.5 of CZ)**.** *$K_{3,3}$ is nonplanar.*

Proof. By the fact that a bipartite planar graph satisfies $m \leq 2n - 4$. □

Exercise. Show that the Petersen graph is nonplanar by using the generalization of Theorem 9.2.

Definition A *subdivision* G' of a graph G is a graph that results from inserting one or more vertices of degree 2 into one or more edges of G.

Theorem (Kuratowski's Theorem)**.** *Graph G is planar if and only if G contains no K_5 or $K_{3,3}$, or subdivision of K_5 or $K_{3,3}$, as a subgraph.*

Exercise. Show that the Petersen graph is nonplanar by using Kuratowski's Theorem.