Bridge Lemma

Student 1

Lemma 1. *Let e be a bridge in a connected graph G joining u to v. Then $G - e$ has exactly 2 connected components, one containing u and the other containing v.*

Proof. We will first show that $G-e$ has ≥ 2 components. For this it suffices to show $G - e$ has u and v in different components. Suppose for the sake of contradiction that u and v are in the same component of $G - e$. In other words, there exists a $u - v$ path P in $G - e$. Path P together with edge e gives a cycle in G, containing e. By Theorem 4.1, e is not a bridge in G, a contradiction.

We will next show that $G - e$ has exactly 2 components. Let x be a vertex in $G - e$. It suffices to show that in $G - e$, vertex x is connected to u, or x is connected to v. By assumption, G is connected, so G contains a $x - u$ path Q. If e is not on Q, then Q is also a path in $G - e$, so x is connected to u in $G - e$. Otherwise e is on Q. In this case, e must be the very last edge on Q, for if e were to appear anywhere else in Q then u would occur more than once, contradicting Q being a path. Therefore, $Q - e$ is an $x - v$ path in $G - e$, so x is connected to v in $G - e$. □