CZ Exercise 4.8

Student 2

Theorem 1. *If every vertex of a graph G has degree at least 2, then G contains a cycle.*

Proof. Suppose G is a graph whose every vertex has degree at least 2. Pick an arbitrary vertex and call it u_0. Since $\deg u_0 \geq 2$, vertex u_0 has a neighbor, say u_1. Since $\deg u_1 \geq 2$, vertex u_1 has a neighbor different from u_0, say u_2.

Since $\deg u_2 \geq 2$, vertex u_2 has a neighbor different from u_1. If u_0 is a neighbor of u_2, then we have a cycle u_0, u_1, u_2, u_0, and we are done. Otherwise, u_2 has another neighbor, say u_3, that is neither u_0 nor u_1.

We continue the reasoning like so. Say we have a path $u_0, u_1, u_2, \ldots, u_i$. We consider u_i. By assumption, $\deg u_i \geq 2$. If some u_j $(0 \leq j < i-1)$ is a neighbor of u_i, we have a cycle $u_j, u_{j+1}, \ldots, u_{i-1}, u_i, u_j$, and we are done. Otherwise, u_i has another neighbor, say u_{i+1}, distinct from any of the u_j $(0 \leq j < i)$, and we extend the path by one more edge to $u_0, u_1, u_2, \ldots, u_i, u_{i+1}$.

But we cannot keep extending the path forever becuase G is finite. So there must exist some i such that $u_0, u_1, u_2, \ldots, u_{i-1}, u_i$ is a path and all (at least 2) neighbors of u_i are some of the vertices u_j $(0 \leq j < i)$. In this case we get a cycle as desired. □