Bridge Lemma proof

Lemma

If e = uv is a bridge in a connected graph G, then G-e has exactly 2 connected components, one containing u and the other containing v.

Proof(by contradiction)

1)In G-e, suppose u and v are in the same connected component H, since H is connected, there must exist some path p from u to v. If p exists in G-e, it must exist in G. With e and p both connecting u and v in G, there exists a cycle containing e between u and v. It contradicts the given condition that e is a bridge in G, so u and v must be in different components.

2)we know G-e must contain at least 2 connected components. When connecting u and v by e, e, as an edge, can at most only join 2 vertices, which means e can only connect the 2 disconnected components that contain u and v. If there are more than 2 disconnected components in G-e, the components without u and v will stay isolated. We know joining u and v by e in G-e will give us G. In the above situation with more than 2 disconnected components in G-e, only components containing u and v will be connected by adding e while other components stay isolated, so G is disconnected, while the given condition says G is connected, so it contradicts. So only can there be exactly 2 disconnected components, one containing u and the other containing v, could satisfy when joining u and v by e will give a connected graph.

Combining 1) and 2), lemma proved.