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Sum Calculus

Indefinite Sum (Antidifference)

In integral calculus we learned about the *indefinite integrals* and the *definite integrals* of real-valued functions. The *indefinite integral* or the *antiderivative* of a function f is defined to be any function F such that $DF = f$. Antiderivatives are not unique. In fact, if F is an antiderivative of f, then any function G defined by $G(x) = F(x) + C$ for all x, where C is any real constant, is also an antiderivative of f. So when we write

$$\int f(x)\,dx = F(x) + C,$$

what we actually mean is that the set of all antiderivatives of f is

$$\{F + C : C \text{ is a constant function}\}$$

where F is some fixed antiderivative of f.

Let's try to define for discrete calculus a concept parallel to the antiderivative. For any real-valued function f, define an *indefinite sum* or *antidifference* of f to be any function F such that $\Delta F = f$. We also write $F = \sum f$ to mean F is an indefinite sum of f.

Lemma. *F and G are antidifferences of f if and only if*

$$G(x) = F(x) + C(x) \quad \textit{for all } x,$$

where C is some function satisfying $C(x + 1) = C(x)$ for all x, i.e., C is a periodic constant function with period 1.

Proof. . . . $\square$

In abused notation, one writes

$$\sum f(x)\,\delta x = F(x) + C(x).$$

Note that in this notation, sigma does not mean taking the sum over a set of numbers like in the sigma notation we have been using. Following tradition, we'll be using this abused notation, and omitting the periodic constant function, for the rest of this section.

The most basic formulas for antidifferences are

Theorem.

$$\begin{aligned}
\sum\,[f(x)+g(x)]\;\delta x &= \sum f(x)\,\delta x + \sum g(x)\,\delta x \\
\sum\,[f(x)-g(x)]\;\delta x &= \sum f(x)\,\delta x - \sum g(x)\,\delta x \\
\sum cf(x)\,\delta x &= c\sum f(x)\,\delta x.
\end{aligned}$$

Proof. . . . $\square$

How do we find an antidifference of a given function f? In the continuous case, we do it by "inverting the theorems for derivatives", e.g., from the theorem $Dx^n = nx^{n-1}$, we get $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$. We'll do the same thing in the discrete case.

Here is a collection of theorems we get by inverting the theorems for antidifferences we already knew.

Theorem.

$$\begin{aligned}
\sum a\,\delta x &= ax \qquad (a \in \mathbb{R}) \\
\sum x^{\underline{m}}\,\delta x &= \frac{x^{\underline{m+1}}}{m+1} \qquad (m \in \mathbb{Z},\ m \neq -1) \\
\sum (ax+b)^{\underline{m}}\,\delta x &= \frac{(ax+b)^{\underline{m+1}}}{a(m+1)} \qquad (a \in \mathbb{R},\ m \in \mathbb{Z},\ a \neq 0,\ m \neq -1) \\
\sum a^x\,\delta x &= \frac{a^x}{a-1} \qquad (a \in \mathbb{R},\ a \neq 1)
\end{aligned}$$

Proof. ... $\square$

Notice that we still can't do $\sum x^{\underline{-1}}\,\delta x = \sum \frac{1}{x+1}\,\delta x$ yet. The situation is similar to the continuous calculus where inverting the formula for Dx^m does not give us an antiderivative for $\frac{1}{x}$. We had to find out as a separate fact that

$$\int \frac{1}{x}\,dx = \ln|x|.$$

What about the function in discrete calculus that is similar to $\ln x$? It turns out that the *Harmonic Numbers* H_n defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is the discrete parallel to the natural log function.

Theorem.

$$\sum \frac{1}{x+1}\,\delta x = H_x.$$

Proof. ... $\square$

We also have the following theorem similar to "integration by parts" of continuous calculus.

Theorem (Summation by parts)**.**

$$\sum f\Delta g = fg - \sum Eg\Delta f$$

Proof. . . . $\square$

Definite Sum and the Fundamental Theorem

The definite integral of a function f from the lower bound a to the upper bound b is written

$$\int_a^b f(x)\,dx$$

and it denotes the area bounded by the curves $x = a$, $y = f(x)$, $x = b$, and $y = 0$. (The precise definition is as the limit of a certain sum.) The Fundamental Theorem of Calculus states that if f is continuous on $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a).$$

Let's define the *definite sum* of the function f from the lower bound a to the upper bound b, written $\sum_a^b f(x)\,\delta x$, to be

$$\sum_a^b f(x)\,\delta x := \sum_{x=a}^{b-1} f(x).$$

Make sure you understand the difference in meanings of the two sigmas used in the above definition.

Theorem (Fundamental Theorem of Discrete Calculus)**.** *If F is the antidifference of f then*

$$\sum_a^b f(x)\,\delta x = F(x)\big|_a^b = F(b) - F(a).$$

Proof. . . . $\square$

Exercises

1. Find the following antidifferences.

   (a) $\sum[x^{\underline{3}} + 3x^{\underline{2}} - 5x^{\underline{1}} + 4]\,\delta x$

   (b) $\sum(3+2x)^{\underline{5}}\,\delta x$

   (c) $\sum(4-3x)^{\underline{-3}}\,\delta x$

   (d) $\sum x(x+3)(x+6)\,\delta x$

   (e) $\sum x^{\underline{-3}}(2x+1)\,\delta x$

   (f) $\sum x^{\underline{n}} a^x\,\delta x$

   (g) $\sum x^n a^{-x}\,\delta x$ for $n = 1, 2, 3$

2. Using sum calculus, prove that

   (a) $a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] = an + \dfrac{dn(n-1)}{2}$

   (b) $a + ar + ar^2 + \cdots + ar^{n-1} = a \cdot \dfrac{r^n - 1}{r-1}$ if $r \neq 1$

   (c) $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$

   (d) $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. Evaluate the following finite sums:

   (a) $2\cdot4\cdot6 + 4\cdot6\cdot8 + \cdots + 16\cdot18\cdot20$.

   (b) $\dfrac{1}{1\cdot2} + \dfrac{1}{2\cdot3} + \cdots + \dfrac{1}{20\cdot21}$

   (c) $\dfrac{1}{1\cdot2\cdot3} + \dfrac{1}{2\cdot3\cdot4} + \cdots + \dfrac{1}{20\cdot21\cdot22}$

   (d) $2\cdot5\cdot8 + 5\cdot8\cdot11 + 8\cdot11\cdot14 + \cdots + 20\cdot23\cdot26$.

4. Evaluate the following finite sums:

   (a) $\sum_{k=0}^{n} k2^k$

   (b) $\sum_{k=0}^{n} k^3 2^k$