San Skulrattanakulchai
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Homework 1

1. Recall that if $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ are any two matrices, then their product $\mathbf{A} \cdot \mathbf{B}$ is

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

   Recall also that we define matrix exponentiation by declaring $\mathbf{A}^1 = \mathbf{A}$ and for all $n \geq 1$, we declare $\mathbf{A}^{n+1} = \mathbf{A} \cdot \mathbf{A}^n$.

   Let $\mathbf{F}$ be the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Show by mathematical induction that

$$\mathbf{F}^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}$$

   for all $n = 1, 2, \ldots,$ where $\langle f_0, f_1, f_2, \ldots, \rangle$ is the Fibonacci sequence.

2. (a) Find a closed-form formula for

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

   by examining the values of the expression for small values of n.

   (b) Prove your formula correct by mathematical induction.

3. Each square of a 1-by-n chessboard is to be colored red, green, or blue so that no two adjacent squares are colored red. Let c_n be the number of ways this can be done. Give a recurrence for c_n. Give all necessary base cases and carefully explain how you derive the formula for the recursive case.

4. An *arithmetic sequence* is a sequence of the form $\langle a, a+d, a+2d, \ldots, a+(n-1)d, \ldots \rangle$. Let $S(n)$ be the sum of the first n terms of an arithmetic sequence.

(a) Give a closed-form formula for $S(n)$.

(b) $-10 - 6 - 2 + 2 + 6 + \cdots + 102 = ?$

5. A *geometric sequence* is a sequence of the form $\langle a, ar, ar^2, \ldots, ar^{n-1}, \ldots \rangle$. Let $T(n)$ be the sum of the first n terms of a geometric sequence.

   (a) Give a closed-form formula for $T(n)$ if $r = 1$.

   (b) Let $r \neq 1$. Show by mathematical induction that
   $$T(n) = a\frac{1 - r^n}{1 - r} \quad \text{or, equivalently} \quad a\frac{r^n - 1}{r - 1}.$$

   (c) $3/5 + 3/25 + 3/125 + \cdots + 3/25^{10} = ?$

6. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the product computed at each step equals $n(n-1)/2$. (Hint: Use strong induction.)