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Difference Calculus

Motivation

Discrete calculus (or Finite Calculus, or Calculus of Finite Differences) is the analogue of continuous calculus. It deals with spaces and processes that are discrete instead of continuous in nature. Its methods involve the same sort of operations on real numbers like those used in the calculus of real variables, but without taking limits. So instead of differentiation, discrete calculus does *differencing*; and instead of integration, discrete calculus does *summing*. The sum operator solves problems like the following quickly: find closed-form formulas for $\sum_{k=1}^{n} k^3$, for $\sum_{x=1}^{m} \sum_{y=1}^{n} (x+y)^2$, or for $\sum_{k=0}^{n} k2^k$. This application to solving sums is our main motivation for studying the subject. Finite difference methods are also used for polynomial interpolation and numerical approximation of differential equations, but we won't spend much time on this latter application.

Functions and Operators

A function f from a *domain* A to a *codomain* B, written $f : A \to B$, is a rule that associates each element $x \in A$ to exactly one element in B. The element of B associated with x is called the *image of x under f* and is written $f(x)$. We usually use f, g, h, etc. to denote functions.

The functions of Finite Calculus may have as its domain some subset of the nonnegative integers $\mathbb{N}$, the integers $\mathbb{Z}$, the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, and they take values in the reals $\mathbb{R}$ or complex numbers $\mathbb{C}$.

Functions on $\mathbb{N}$ are called *sequences* and we usually use subscript notation to denote them, e.g., $\langle a_0, a_1, a_2, \ldots \rangle$, or $\{a_n\}_{n=0}^{\infty}$, or $\{a_n\}_0^{\infty}$, or even $\{a_n\}$.

Two functions f and g are called *equal*, written $f = g$, if they have the same domain and $f(x) = g(x)$ for all x in the common domain. Equality of functions is an equivalence relation. (Prove!)

The *composition of f and g*, written $f \circ g$, is a function defined by $(f \circ g)(x) = f(g(x))$ for all x. Function composition is associative, but is usually not commutative. (Prove!)

The function 1 (or I) is the *identity function* if $1(x) = x$ for all x.

A function g is called the *inverse* of f if $f \circ g = g \circ f = 1$. we write f^{-1} for the inverse of f. A function f has an inverse function iff f is bijective, i.e., 1-1 and onto.

Define $f^0 = f$ and for all nonnegative integer n, define $f^{n+1} = f \circ f^n$. If f has an inverse, we can define a function to a negative exponent by declaring $f^{-n-1} = f^{-1} \circ f^{-n}$ for all negative integer n. Function exponentiation follows the usual rules of exponents, e.g., $f^m \circ f^n = f^{m+n}$. Moreover, the set $\{f^n : n \in \mathbb{Z}\}$ commutes under composition.

We define *addition*, *subtraction*, *multiplication*, *division*, *exponentiation*, *logarithm*, etc. of real-valued (complex-valued) functions pointwise, e.g., $(f+g)(x) = f(x)+g(x)$ for all x, etc.

Therefore, $+$, $-$, $*$, $/$, etc. of real-valued (complex-valued) functions obey the same laws as the real (and complex) numbers, e.g., $+$ and $*$ are associative, commutative; $*$ distributes over $+$, etc. We use the same symbols for corresponding elements and operations, e.g., 0 is the identity function for $+$, and 1 is the identity function for $*$. In fact, for any $c \in \mathbb{R}$, the symbol c denotes the constant function where $c(x) = c$ for all x.

We define *scalar multiplication* of a function f by a scalar α by $(\alpha f)(x) = \alpha \cdot f(x)$ for all $x \in \mathbb{R}$. Under function addition and scalar multiplication, the set of real-valued functions forms a vector space.

Difference Operator

For our purpose, we define an *operator* to be a function whose domain and codomain are sets of real-valued functions. Later we will learn that discrete calculus is interesting when the functions operated by these operators are sequences. Here are some example operators.

In continuous calculus, we studied the *differential operator* D or $\frac{d}{dx}$ defined as

$$Df(x) = \frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{for all } x$$

for any differentiable real-valued function f. In this section we study an operator Δ of discrete calculus that corresponds to the D operator of continuous calculus.

The *shift operator* E maps a function f to the function Ef, where $Ef(x) = f(x+1)$ for all x in the domain of f.

Recall that the *identity operator* 1 (or I) is such that $1f = f$ for every function f.

The *forward difference operator* Δ is defined to be $\Delta = E - I$, i.e.,

$$\Delta f(x) = f(x+1) - f(x)$$

for all function f and all x in the domain of f. Intuitively, in the definition for derivative, we let h approach 0 discretely, through positive integral values, and the closest that h can get to 0 without being equal to it is 1.

Theorem. *(i)* $\Delta(f \pm g) = \Delta f \pm \Delta g$

(ii) $\Delta(\alpha \cdot f) = \alpha \cdot \Delta f$

(iii) $\Delta(f \cdot g) = Ef \cdot \Delta g + \Delta f \cdot g = \Delta f \cdot Eg + f \cdot \Delta g$

(iv) $\Delta(f/g) = \dfrac{\Delta f \cdot g - f \cdot \Delta g}{g \cdot Eg}$ *where* $g, Eg \neq 0$

Proof. . . . □

An operator A is *linear* if the following two conditions are satisfied:

(i) $A(f+g) = A(f) + A(g)$ for all functions f, g.

(ii) $A(\alpha f) = \alpha A(f)$ for all scalar α and function f.

One corollary of the above theorem is that Δ is a linear operator.

Notational Shorthand

Statements like $\frac{d}{dx}x^2 = 2x$ or $Dx^2 = 2x$ in differential calculus is an abuse of notation. What they actually say is that if f is a function defined by $f(x) = x^2$ for all x, then its derivative Df is the function defined by $Df(x) = 2x$ for all x. However, this abused notation is so widespread that we'll go along with it. Here is a theorem stated in this abused notation.

Theorem. *(i)* $\Delta c = 0$ *for any* $c \in \mathbb{R}$

(ii) $\Delta^m x^m = m!$ *for nonnegative integer* m

(iii) $\Delta^m x^n = 0$ *for nonnegative integers* m, n *with* $m > n$

Proof. . . . □

Falling Factorial Powers

In differential calculus we have $Dx^m = mx^{m-1}$. Computing, $\Delta x^2 = (x+1)^2 - x^2 = 2x + 1 \neq 2x$, we find that the formula for D does not carry over to Δ. However, similar formula holds for another kind of power function.

Let m be a positive integer. For any $x \in \mathbb{R}$, define the mth *falling factorial power* of x, written $x^{\underline{m}}$, and read *x to the m falling*, to be

$$x^{\underline{m}} = \overbrace{x(x-1)\cdots(x-m+1)}^{m \text{ factors}}.$$

We define $x^{\underline{0}} = 1$. Note that the familiar factorial function $n!$ can be written $n^{\underline{n}}$.

Theorem. *For nonnegative integer m, we have $\Delta x^{\underline{m}} = mx^{\underline{m-1}}$.*

Proof. ... ∎

We know that $x^{m+n} = x^m x^n$. What's the corresponding formula for falling powers? Consider the expression $x^{\underline{m+n}}$, where m, n are nonnegative integers. It's clear that

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}.$$

We would like to extend the definition of the falling powers to negative integer exponents so that the above formula holds. Plugging in $m = -n$ in the above formula gives us

$$x^{\underline{-n}} = \frac{1}{(x+n)^{\underline{n}}} = \frac{1}{(x+1)(x+2)\cdots(x+n)}$$

for positive integer n. We will take this to be the definition of falling negative powers. Using this definition, we find that $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$ for all integers m, n. (Prove!) Even nicer, the last theorem generalizes.

Theorem. *For any integer m, we have $\Delta x^{\underline{m}} = mx^{\underline{m-1}}$.*

Proof. ... ∎

For a function f and integer m, define $f(x)^{\underline{m}}$ to be

$$f(x)^{\underline{m}} = \begin{cases} f(x)f(x-1)\cdots f(x-m+1) & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ \dfrac{1}{f(x+1)f(x+2)\cdots f(x-m)} & \text{if } m < 0. \end{cases}$$

Theorem. *For any integer m and any reals a, b, we have $\Delta(ax+b)^{\underline{m}} = am(ax+b)^{\underline{m-1}}$.*

Proof. ... □

Exercise *Rising factorial power*, $x^{\overline{m}}$, is a parallel concept to falling factorial power. Define rising factorial power and find interesting theorems about it.

Polynomials and Stirling Numbers

From the definition of falling factorial powers, we see that every falling power is a polynomial like so

$$\begin{aligned} x^{\underline{1}} &= x \\ x^{\underline{2}} &= x^2 - x \\ x^{\underline{3}} &= x^3 - 3x^2 + 2x \\ x^{\underline{4}} &= x^4 - 6x^3 + 11x^2 - 6x \\ x^{\underline{5}} &= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x \end{aligned}$$

etc. and using the above identities, we can also write every ordinary power as a polynomial of falling powers like so

$$\begin{aligned} x &= x^{\underline{1}} \\ x^2 &= x^{\underline{2}} + x^{\underline{1}} \\ x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\ x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\ x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}}. \end{aligned}$$

Theorem. *Every ordinary polynomial has a unique representation as a polynomial in falling factorial powers. Every polynomial in falling factorial powers has a unique representation as an ordinary polynomial.*

Proof. . . . □

The interesting question is: how do we find the coefficients in the representation of these powers? We will cover this in class.

The coefficients of the polynomials in the first set of identities are called *Stirling numbers of the first kind* (`http://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind`).

The coefficients of the polynomials of falling powers in the second set of identities are called *Stirling numbers of the second kind* (`http://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind`).

Exponential Functions

In differential calculus we learned that $De^x = e^x$. Technically we say that the function $x \mapsto e^x$ is a *fixed-point* of the D operator. What is the corresponding formula in discrete calculus? In other words, does the Δ operator have a fixed-point? The answer is yes, the function $x \mapsto 2^x$ is a fixed-point.

Theorem. *(i)* $\Delta c^x = (c-1)c^x$ *for all* $c \in \mathbb{R}$.

(ii) $\Delta 2^x = 2^x$.

Proof. . . . □

The Discrete Taylor Formula

Lemma. *$E^n f(x) = f(x+n)$ for all $n \in \mathbb{Z}$. In particular, $E^n f(0) = f(n)$.*

Proof. ... □

Lemma. *The operators E, I, and Δ commute under opertor composition.*

Proof. ... □

Theorem. *For all nonnegative integer n,*

$$E^n = (\Delta + I)^n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k$$

Proof. The result holds as a result of the Binomial Theorem and that Δ and I commute under composition. □

Corollary. *For all function f and all nonnegative integer n,*

$$f(n) = (\Delta + I)^n f(0) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0).$$

Proof. Applying the above theorem to f and evaluating the result at 0, we have

$$E^n f(0) = (\Delta + I)^n f(0) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0).$$

By the above lemma $E^n f(0) = f(n)$. The Corollary results. □

Gregory-Newton forward difference interpolation formula

If we know the values of $\Delta^k f(0)$ for all $k = 0, 1, 2, \ldots, n$, we can use the discrete Taylor formula to find a polynomial of degree at most n that interpolates f as follows. Define the polynomial P_n by declaring

$$P_n(x) := \sum_{k=0}^{n} \binom{x}{k} \Delta^k f(0) = \sum_{k=0}^{n} \frac{\left(\Delta^k f(0)\right) x^{\underline{k}}}{k!}$$

for all real x. Then P_n is a polynomial of degree at most n that interpolates the function f at $x = 0, 1, \ldots, n$, i.e., $P_n(x) = f(x)$ for all $x = 0, 1, \ldots, n$.

Another application of the fact that $\Delta = E - I$ and that these 3 operators Δ, E, and I commute is the calculation of $\Delta^n f(x)$ in terms of a sum of $f(\cdot)$'s. We have

$$\Delta^n = (E - I)^n = \sum_{k=0}^{n} \binom{n}{k} E^{n-k}(-I)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{n-k}.$$

For example,

$$\Delta^3 f(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x).$$