San Skulrattanakulchai
Gustavus Adolphus College
March 25, 2014

Closure Properties of CFLs

Sipser Ch 2: Exercises 2.2, 2.15, 2.16, 2.18

Theorem. *The CFLs are closed under* $\cup$, $\circ$, *and* *. □

Theorem. *If* P *is a PDA and* D *is a DFA, then* $L(P) \cap L(D)$ *is a CFL.*

Proof. Construct a PDA that simultaneously simulates P and D. □

Question. Why does this construction fail when both P and D are PDA's?

In the rest of this handout let $\Sigma = \{a, b, c\}$.

Lemma. *Let*

$$E = \{\, w \in \Sigma^* : |w|_a = |w|_b = |w|_c \,\}.$$

E is not a CFL.

Proof. Suppose it were. Then $F = E \cap L(a^*b^*c^*)$ is a CFL. But $F = \{\, a^n b^n c^n : n \geq 0 \,\}$ and we have already seen that F is not a CFL, a contradiction. □

Theorem. *The CFLs are not closed under intersection.*

Proof. This is because E is the intersection of 2 CFLs:

$$E = \{\, w \in \Sigma^* : |w|_a = |w|_b \,\} \cap \{\, w \in \Sigma^* : |w|_b = |w|_c \,\}.$$

To show the RHS languages are CFL's, use PDAs. □

Theorem. *The CFLs are not closed under complement.*

Proof. E is the complement of the CFL

$$\overline{E} = \{\, w \in \Sigma^* : |w|_a \neq |w|_b \,\} \cup \{\, w \in \Sigma^* : |w|_b \neq |w|_c \,\}.$$

To show the RHS languages are CFL's, note that $i \neq j$ means $i < j$ or $i > j$.

Another proof is to note that the language $\mathtt{SQ} = \{\, ww : w \in \{0,1\}^* \,\}$ is not context-free; however, its complement is. □