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Logic

A *proposition* is a statement that is either *true* or *false*, but not both. Here are some example propositions.

- $\sqrt{2}$ is an irrational number.
- $1 < 1$.
- Some integers x, y, z satisfy $x^3 + y^3 = z^3$.
- There are no positive integers x, y, z such that $x^4 + y^4 = z^4$.

And here are some example non-propositions.

- What do you mean by that?
- How pretty!
- Please give me a call.
- Barack Obama is an honest president.
- 23 is an interesting number.
- $x^2 + 3x - 4 = 0$.
- This statement is false.

A *compound proposition* can be constructed from any propositions p, q using the *logical connectives* $\wedge$, $\vee$, $\neg$, $\implies$, $\iff$.

Compound Proposition	**Formal Sentence**	**Meaning**
conjunction	$p \wedge q$	p and q
disjunction	$p \vee q$	p or q
negation	$\neg p$	not p
implication	$p \implies q$	p implies q
biconditional	$p \iff q$	p if and only if q

In the implication $p \implies q$, the proposition p is called *premise* or *hypothesis* or *antecedent*, and the proposition q is called *conclusion* or *consequence* or *consequent*.

Truth Tables

...

These are some of the ways to express the implication "$p \implies q$" in English.

if p **then** q

p **implies** q

q **is implied by** p

p **only if** q

q **if** p

p **is sufficient for** q

q **is necessary for** p

q **whenever** p

These are some of the ways to express the biconditional "$p \iff q$" in English.

p **if and only if** q

p **is necessary and sufficient for** q

q **is necessary and sufficient for** p

p **implies** q **and vice versa**

if p **then** q **and vice versa**

...

Definitions:

- An implication is said to be *trivially true* when its conclusion is true.

- An implication is said to be *vacuously true* when its hypothesis is false.
- The *converse* of "$p \implies q$" is "$q \implies p$."
- The *contrapositive* of "$p \implies q$" is "$\neg q \implies \neg p$."
- Two propositions are *logically equivalent* when they have the same truth value for all possible combinations of the truth values of their constituent propositions.
- A *tautology* is a proposition that's always true, eg, "$p \vee \neg p$."
- A *contradiction* is a proposition that's always false, eg, "$p \wedge \neg p$."
- A *contingency* is proposition that's sometimes true and sometimes false, eg, "$p \vee p$."

Question: which of these are tautology? contradiction? contingency?

$p \implies p$

$(\neg p) \implies p$

$[(p \implies q) \wedge p] \wedge (\neg q)$

$[p \vee (p \wedge q)] \implies p$

$[p \wedge (p \vee q)] \iff \neg p$

Some important logical equivalences

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge T \equiv p$	$p \vee F \equiv p$
$p \wedge F \equiv F$	$p \vee T \equiv T$
$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$	$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$
$\neg(\neg p) \equiv p$	$p \iff q \equiv q \iff p$
$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
$p \implies q \equiv (\neg q) \implies (\neg p)$	$p \implies q \equiv (\neg p) \vee q$
$\neg(p \implies q) \equiv p \wedge (\neg q)$	$(\neg p) \implies F \equiv p$
$p \iff q \equiv (p \implies q) \wedge (q \implies p)$	$p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$(p \vee q) \implies r \equiv (p \implies r) \wedge (q \implies r)$	$(p \wedge q) \implies r \equiv p \implies (q \implies r)$

Non-proposition statements containing variables, the so-called *open sentences*, can be turned into propositions by quantifying the variables.

The *existential quantifier* $\exists$ means "at least one." In English, we can express it like

there is ...	there exists ...	some ...	for some ...

The *universal quantifier* $\forall$ means "all." Some ways to say it in English are

all ...	for all ...	every ...	for every ...
each ...	for each ...	any ...	for any ...

Question: What do the following English sentences mean?

- There is a male student in the MCS-236 class.
- There is a female student in the MCS-236 class.
- A rational number is not integral.
- Odd integers are multiples of 3.
- Some schiffs are fibs.
- All fibs are fobs.
- $a + b = b + a$.

Exercise: Rewrite all ambiguous sentences above to disambiguate them.

Here are two common conventions in mathematics writing.

- Universal quantifiers are routinely omitted when possible.
- In definition, the word 'if' actually means 'if and only if.'

Beware of the following.

- Depending on context, the word 'one' could mean 'at least one' or 'exactly one.' Similarly for 'two', 'there', ..., etc.
- Oftentimes the word 'a' should be replaced by 'any.'

To negate quantified propositions, use the logical equivalence

$$\neg[\forall x.\varphi(x)] \equiv \exists x.\neg\varphi(x)$$

or the logical equivalence

$$\neg[\exists x.\varphi(x)] \equiv \forall x.\neg\varphi(x)$$

A sentence of the form $\forall x \in X.\varphi(x)$ is short for $\forall x[x \in X \implies \varphi(x)]$.

A sentence of the form $\exists x \in X.\varphi(x)$ is short for $\exists x[x \in X \wedge \varphi(x)]$.

A sentence of the form, "For every element of the nonempty set ... blah ..." is always true.

A sentence of the form, "There exists some element in the nonempty set such that ... blah ..." is always false.

The term *unique* in mathematical writing means "exactly one."

The term *distinct* in mathematical writing means "different."

The term *write* in mathematical writing often means "let."

The phrase *without loss of generality* in mathematical writing has many meanings, as will be demonstrated throughout this course!