MCS-228: Proofs in Mathematics and Computer Science Handout #C1
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Solving Quadratic Equations

Problem Given real numbers a, b, and c, where $a > 0$, find all real numbers x such that $ax^2 + bx + c = 0$. (Do you see why we may assume $a > 0$ without losing any generality?)

Answer First we assume x can be any complex number. The equation

$$ax^2 + bx + c = 0$$

holds if and only if the equation

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

holds if and only if the equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

holds if and only if the equation

$$x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

holds if and only if the equation

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

holds if and only if the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

holds if and only if the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

holds if and only if

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

holds if and only if

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

holds if and only if

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Thus, there are exactly 2 (possibly non-unique) solutions in complex numbers:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are no real solutions when $b^2 - 4ac < 0$. There is exactly 1 real solution when $b^2 - 4ac = 0$. There are exactly 2 real solutions when $b^2 - 4ac > 0$. $\square$