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Propositions 1.43a and 1.43b

1.43. Proposition Elementary consequences of the field axioms.

a) $x + z = y + z$ implies $x = y$

b) $x \cdot 0 = 0$

Proof. a) Let x, y, and z be real numbers satisfying $x+z = y+z$. By axiom A4, let w be a real number such that $z + w = 0$. Adding w to both sides of equation $x + z = y + z$ leads to

$$(x + z) + w = (y + z) + w$$

which by axiom A1 leads to

$$x + (z + w) = y + (z + w)$$

which by definition of w leads to

$$x + 0 = y + 0$$

which by axiom A3 leads to

$$x = y,$$

the desired result.

b) Let x be any real number. Applying axiom A3 to the case when $x = 0$ we have $0 + 0 = 0$. Therefore,

$$x \cdot (0 + 0) = x \cdot 0,$$

which by axiom DL leads to

$$x \cdot 0 + x \cdot 0 = x \cdot 0.$$

By axiom M0, $x \cdot 0$ is a real number. By axiom A4, let w be a real number such that $x \cdot 0 + w = 0$. Adding w to both sides of equation $x \cdot 0 + x \cdot 0 = x \cdot 0$ gives

$$(x \cdot 0 + x \cdot 0) + w = x \cdot 0 + w$$

which by axiom A1 leads to

$$x \cdot 0 + (x \cdot 0 + w) = x \cdot 0 + w$$

which by definition of w leads to

$$x \cdot 0 + 0 = 0$$

which by axiom A3 leads to

$$x \cdot 0 = 0,$$

the desired result.

$\square$