San Skulrattanakulchai
Gustavus Adolphus College

The Penny Problem

Definition. A *bag* or *multiset* is a collection of elements that may contain any element multiple times. We use the same notation for bags as for sets, and define membership, size, bag containment, etc., exactly the same way as for sets. For example, the bag $A = \{1, 2, 1, 3\}$ has four elements; it contains 1 twice, 2 once, and 3 once. If $B = \{1, 1, 2, 2, 3, 4\}$, then $B \supseteq A$. And so on and so forth.

The Penny Problem. Given a bag A of n natural numbers, we create a new bag B by first going through every number x in bag A and putting $x - 1$ into bag B if $x - 1 > 0$, and then putting n into bag B. For example, applying the operation to bag $\{1, 1, 2, 5\}$ results in the bag $\{1, 4, 4\}$. Find all bags A such that applying this operation leaves them unchanged.

Answer. The answer is in this theorem.

Theorem. *A bag A is unchanged under this operation if and only if $A = [n]$ for some natural number n.*

Proof. First suppose $A = [n]$, where n is a natural number. The new bag B satisfies $B \supseteq [n - 1]$ since by the first step of the operation each integer from 2 to n that was in bag A was reduced by one and put into bag B. Also, $n \in B$ by the second step of the operation. Thus $B = [n]$. So A is unchanged under the operation.

Now let A be a bag that is unchanged under this operation, i.e., bag B that results from this operation equals bag A. We claim that

(i) for all $x \in A$, if $x > 1$ then $x - 1 \in A$, and

(ii) $n \in A$.

Part (i) holds since the first step of the operation subtracts one from each number in bag A and puts the resulting number into bag B (which equals A) if it is positive. Part (ii) holds since the second step of the operation puts n into bag B (which equals A).

Starting from the truth of Part (ii) of the claim and applying Part (i) leads to $n-1 \in A$, then applying Part (i) for $x = n-1$ leads to $n-2 \in A$, etc., all the way until $x = 2$. Thus we conclude that $A \supseteq [n]$. But $|A| = n$ by assumption and $[n]$ already has size n. Thus, A does not contain any other numbers beside all the positive integers from 1 to n, once each. Hence, $|A| = [n]$. $\square$