San Skulrattanakulchai
Gustavus Adolphus College

Induction

Theorem. *Any natural number n satisfies*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

Proof by Weak Induction. When $n = 0$, the left hand side of Equation (1) equals 0 and the right hand side equals $\frac{0 \cdot 1}{2} = 0$. So Equation (1) holds in the base case.

Now let k be any nonnegative integer and assume inductively that Equation (1) holds when $n = k$. We have

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

so Equation (1) holds when $n = k + 1$ as well.

The result follows by Mathematical Induction. □

Theorem. *For any integer n, if $n \geq 4$, then $2^n \geq n^2$.*

Proof by Weak Induction. When $n = 4$, we have $2^4 = 16 \geq 16 = 4^2$ so the inequality holds in the base case.

Now let n be any integer ≥ 4 and assume inductively that $2^n \geq n^2$. Since $n \geq 4 \geq 3$, we have

$$n - 1 \geq 2 \geq \sqrt{2},$$

whence

$$(n-1)^2 \geq 2$$

whence

$$n^2 - 2n + 1 \geq 2$$

whence

$$n^2 \geq 2n + 1.$$

Therefore,

$$2^{n+1} = 2^n + 2^n \geq n^2 + n^2 \geq n^2 + 2n + 1 = (n+1)^2.$$

Therefore, $2^n \geq n^2$ whenever $n \geq 4$ by Mathematical Induction. $\square$

Theorem. *Any positive integer ≥ 2 can be written as a product of primes.*

Proof by Strong Induction. The integer 2 is prime so the result holds in the base case. Now let n be any integer > 2 and assume inductively that all integers ≥ 2 but $< n$ can be written as a product of primes. If n itself is prime, then we are done. So suppose from now on that n is not prime. So there exists an integer d_1, where $1 < d_1 < n$, such that $n = d_1 d_2$ for some integer d_2. We claim that d_2 satisfies $1 < d_2 < n$ as well. That $1 < d_2$ follows from $d_1 < n = d_1 d_2$ and $d_1 > 0$. To see that $d_2 < n$, assume for the sake of contradiction that $d_2 \geq n$. Since $d_1 > 1$, we have $n = d_1 d_2 > n$, a contradiction. By inductive hypothesis, it follows that both d_1 and d_2 are product of primes, say $d_1 = p_1 \cdot p_2 \cdots p_k$ and say $d_1 = q_1 \cdot q_2 \cdots q_\ell$ where the p's and q's are primes. So $n = d_1 d_2 = p_1 \cdot p_2 \cdots p_k \cdot q_1 \cdot q_2 \cdots q_\ell$, a product of primes as desired.
The result follows by Mathematical Induction. $\square$