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Proving equality of sets

Exercise 2.51a Using statements about membership, prove

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

where A, B, C are any sets.

Proof. We first show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Let $x \in A \cup (B \cap C)$. Thus, $x \in A$ or $x \in B \cap C$. It's always true that either $x \in A$ or $x \notin A$. Let's consider both possibilities.
Suppose $x \in A$. Then $x \in A$ or $x \in B$. Also, $x \in A$ or $x \in C$. Thus $x \in A \cup B$ and $x \in A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$.
Now suppose $x \notin A$. This supposition together with the assumption $x \in A \cup (B \cap C)$ implies $x \in B \cap C$, that is, $x \in B$ and $x \in C$. Since $x \in B$, it follows that $x \in A$ or $x \in B$, in other words, $x \in A \cup B$. Since $x \in C$, it follows that $x \in A$ or $x \in C$, in other words, $x \in A \cup C$. Therefore, $x \in A \cup B$ and $x \in A \cup C$, that is, $x \in (A \cup B) \cap (A \cup C)$.
Either way we have $x \in (A \cup B) \cap (A \cup C)$.

We next show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Let $x \in (A \cup B) \cap (A \cup C)$. Thus, $x \in A \cup B$ and $x \in A \cup C$. I It's always true that either $x \in A$ or $x \notin A$. Let's consider both possibilities.
Suppose $x \in A$. Then $x \in A$ or $x \in B \cap C$, that is, $x \in A \cup (B \cap C)$.
Now suppose $x \notin A$. Since $x \in A \cup B$, we have $x \in A$ or $x \in B$. Hence, $x \in B$. Since $x \in A \cup C$, we have $x \in A$ or $x \in C$. Hence, $x \in C$. Therefore, $x \in B$ and $x \in C$, that is, $x \in B \cap C$. Thus, $x \in A$ or $x \in B \cap C$, that is, $x \in A \cup (B \cap C)$.
Either way we have $x \in A \cup (B \cap C)$. □