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Cardinality of the union of two countable sets

Theorem. *If sets A and B are countable, then $A \cup B$ is countable.*

Proof. By countability of A, let $a_1, a_2, \ldots,$ enumerate A, that is, it lists each element of A exactly once. By countability of B, let $b_1, b_2, \ldots,$ enumerate B. Obtain the sequence $c_1, c_2, \ldots,$ by defining

$$c_n = \begin{cases} a_{2n-1}, & \text{if } n \text{ is odd} \\ b_{2n}, & \text{if } n \text{ is even.} \end{cases}$$

From the sequence $c_1, c_2, \ldots,$ we obtain a subsequence $d_1, d_2, \ldots,$ that enumerates $A \cup B$ as follows. Set $d_1 := c_1$. Suppose d_n has already be chosen. We choose d_{n+1} by the following method. The set $X_n = \{\, k \in \mathbb{N} : c_k$ is not equal to d_i for any $i = 1, 2, \ldots, n \,\}$ is not empty since the subsequence $c_1, c_3, c_5, \ldots,$ of $\langle c \rangle$ enumerates A and is infinite but $\langle d_1, d_2, \ldots, d_n \rangle$ is finite. By the Well-Ordering property of $\mathbb{N}$, let k be the least element of X_n, and set $d_{n+1} := c_k$.

Clearly $d_1, d_2, \ldots,$ lists every element of $A \cup B$, each exactly once. Thus $A \cup B$ is either finite or countable. However, A was countable by assumption and $A \subseteq A \cup B$, so $A \cup B$ is not finite; thus it is countable. $\square$