UVa 116 - Unidirectional TSP

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Unidirectional TSP

Problem

Given an `m` x `n` matrix of integers, you are to write a program that computes a path of minimal weight. A path starts anywhere in column 1 (the first column) and consists of a sequence of steps terminating in column `n` (the last column). A step consists of traveling from column `i` to column `i+1` in an adjacent (horizontal or diagonal) row. The first and last rows (rows 1 and `m`) of a matrix are considered adjacent, i.e., the matrix "wraps" so that it represents a horizontal cylinder.

The *weight* of a path is the sum of the integers in each of the `n` cells of the matrix that are visited.

Solution by Dynamic Programming

Let A be the given `m x n` matrix of integers. For all i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$, define $w(i,j)$ to be the weight of a lightest "path" starting in the cell at row i and column j and ending in some cell in column n.

A "path" is defined like in the problem statement, that is, it consists of steps, where a step consists of traveling from column `i` to column `i+1` in an adjacent (horizontal or diagonal) row.

Recurrence

For all $1 \le i \le m$

- $w(i, n) = A_{i,n}$
- For all $1 \le j < n$,
  $w(i, j) = A_{i,j} + \min\{w(i-1, j+1), w(i, j+1), w(i+1, j+1)\}$

where the arithmetic on i is done modulo m.

We are seeking $\min\{w(i, 1) : 1 \le i \le m\}$. Moreover, the solution has to be the lexicographically smallest.

DP Tables

For this problem it is convenient to also compute a companion minimizer table. So for all $1 \leq i \leq m$ and $1 \leq j < n$ let's define $z(i,j)$ to be the ordered pair $(i', j+1)$ where i' is the the smallest index achieving the minimum in the recursive case of the definition for w.

Implementation of the Algorithm

- Step 1. Fill in the two tables $w(\cdot, \cdot)$ and $z(\cdot, \cdot)$, making sure to let the column index be decreasing. The row index can be in any order.
- Step 2. Compute $\min\{w(i, 1) : 1 \leq i \leq m\}$ and let i^* be a minimizer of smallest value.
- Step 3. Output the answer starting from cell $(i^*, 1)$ and using the $z(\cdot, \cdot)$ table to determine the rest of the path.

Running Time

- Step 1 takes time $O(mn)$.
- Step 2 takes time $O(m)$.
- Step 3 takes time $O(n)$.

So total running time is $O(mn)$.