Impartial Combinatorial Games

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Combinatorial Game Theory

A **combinatorial game** has these properties:

- there are two players
- players alternate turn making a move
- the players have perfect information about the current state of the game
- there is no chance element in the game, i.e., no card deals, no dice or coin flips, etc
- the game has a finite number of positions (game states)
- the game always ends, i.e., it is acyclic
- the game ends when a player can't make a move
  - In a **normal-play** game, the player who makes the last move wins.
  - In a **misère-play** game, the player who makes the last move loses.

Strategy

Playing perfectly means if a player can force a win, then she always makes a move that allows her to force a win; and if she cannot force a win, then she simply makes any move that is available.

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`Who wins the game if both players play perfectly?`

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`Who wins the game if both players play perfectly?`

In theory, we assume perfect play on both players.

Types of combinatorial games

- In an **impartial game**, the set of allowable moves for each position does not depend on which player is moving. Example impartial games are Tic Tac Toe, Nim, Bachet's game.
- In a **partizan game**, the set of allowable moves for each position depends on who is making the move. Example partizan games are Chess, Checkers, GO.

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From now on we will talk about impartial games only.

Theory of impartial games

Outcome Classes

- A $\mathcal{P}$-position secures a win for the previous player. In other words, it is a losing position for the player whose turn is to move. An example $\mathcal{P}$-position in a game of Nim is where exactly two piles of stones are left and both piles have equal number of stones.
- An $\mathcal{N}$-position secures a win for the next player. An example $\mathcal{N}$-position in a game of Nim is where only one pile of stones remains.

A **terminal position** is a position where no move is available.

Outcome of an Impartial Game

Theorem

An impartial game's position is either a $\mathcal{P}$-**position** or an $\mathcal{N}$-**position**.

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Theorem

Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with the properties:

- every option of a position in A is in B
- every position in B has at least one option in A

Then A is the set of $\mathcal{P}$-positions and B is the set of $\mathcal{N}$-positions.

Algorithm

To find an outcome of an impartial game:

1. Draw the game tree.
2. Starting from the leaves, recursively label the vertices, using either $\mathcal{P}$ or $\mathcal{N}$ labels, using the recursive definition of $\mathcal{P}$-position and $\mathcal{N}$-position as given in the previous theorem.

Note: Every terminal position is a $\mathcal{P}$-position.