MCS 220 Lecture Proofs

Tom LoFaro

November 9, 2009

Lecture 9/15/09: *For all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|1 - x| < \delta$ then $|2x + 3 - 5| < \epsilon$.*

Proof: Fix $\epsilon > 0$. Let $\delta \leq \epsilon/2$. If

$$1 - \delta < x < 1 + \delta$$

then

$$1 - \frac{\epsilon}{2} < x < 1 + \frac{\epsilon}{2}.$$

If we multiply this inequality by 2 we get

$$2 - \epsilon < 2x < 2 + \epsilon.$$

Finally, add 3 to get

$$5 - \epsilon < 2x + 3 < 5 + \epsilon.$$

□

Lecture 9/18/09: *For all $n \in \mathbb{N}$, $7^n - 6n - 1$ is divisible by 36.*

Proof: We will proceed by induction. If $n = 1$ then $7^n - 6n - 1 = 0$ which is divisible by 36 since 0 is divisible by all integers.

Assume that $7^n - 6n - 1$ is divisible by 36. This implies that there exists $k \in \mathbb{N}$ such that $7^n - 6n - 1 = 36k$. To prove the statement for the $n + 1$ case we note that

$$7^{n+1} - 6(n + 1) - 1 = 7^{n+1} - 6n - 7 = 7^{n+1} - 42n - 7 + 36n.$$

Factoring out a 7 from the first three terms we get

$$7^{n+1} - 6(n + 1) - 1 = 7(7^n - 6n - 1) + 36n = 7(36k) + 36n = 36(7k + n).$$

Since k and n are both integers $7k+n$ is an integer and hence $7^{n+1}-6n-7$ is divisible by 36 as desired. $\square$

Lecture 9/29/09: $\lim_{n\to\infty} \frac{3n+10}{n+2} = 3.$

Proof: Fix $\epsilon > 0$ and let $N > 4/\epsilon - 3$. If $n > N$ then $n > 4/\epsilon - 2$ and hence

$$\frac{4}{n+2} < \epsilon.$$

This implies that

$$\frac{4}{n+2} + \frac{3(n+2)}{n+2} - 3 = \frac{3n+10}{n+2} - 3 < \epsilon.$$

Since the left hand side of this inequality is positive for $n > 0$ we get

$$\left|\frac{3n+10}{n+2} - 3\right| < \epsilon$$

as desired. $\square$

Lecture 10/22/09: *The function* $f(x) = 2x^2 + 1$ *is continuous.*

Proof: Fix x_0 and let (x_n) be any sequence such that (x_n) converges to x_0. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 2x_n^2 + 1 = 2x_0^2 + 1 = f(x_0).$$

The second equality follows from the properties of convergent sequences. Therefore $f(x)$ is continuous. $\square$

Lecture 10/23/09: *The function*

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

is discontinuous at $x_0 = 0$.

Proof: Let $x_n = -1/n$ so that $\lim_{n\to\infty} x_n = 0$. Then $f(x_n) = -1$ for all n. Thus

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} -1 = -1 \neq f(0).$$

Therefore f is discontinuous at 0. $\square$

Lecture 10/23/09: *The function*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x_0 = 0$.

Proof: Let (x_n) be any sequence such that (x_n) converges to 0. Since $-1 \leq \sin(x) \leq 1$ for all x it follows that

$$-x_n \leq x_n \sin\left(\frac{1}{x_n}\right) \leq x_n.$$

Since both sequences (x_n) and $(-x_n)$ converge to 0

$$\lim_{n\to\infty} x_n \sin\left(\frac{1}{x_n}\right) = 0$$

and thus f is continuous at $x = 0$. $\square$

Lecture 11/1/09: *The function* $f(x) = x^{-2}$ *is* not *uniformly continuous on the interval* $(0, 1)$.

Proof: We will show that for every $\delta > 0$ there exists x, $y \in (0, 1)$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| > 1$.

Assume that $\delta < 1/2$ and let $x = \delta$ and $y = \delta + \delta/2 = 3\delta/2$. Then

$$|f(x) - f(y)| = \frac{1}{\delta^2} - \frac{4}{9\delta^2} = \frac{5}{9\delta^2}.$$

Since we have assumed that $\delta < 1/2$ we have that that

$$|f(x) - f(y)| = \frac{5}{9\delta^2} \geq \frac{5}{9(\frac{1}{2})^2} = \frac{20}{9} > 1.$$

Therefore $f(x)$ is not uniformly continous on the interval $(0, 1)$. $\square$

Lecture 11/9/09: *If* (f_n) *converges uniformly to a function* f *on* S *and* f_n *is continous on* S *for all* n *then* f *is continuous on* S.

Proof: Fix $\epsilon > 0$ and $x_0 \in S$. By the triangle inequality

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

Since f_n converges uniformly to f on S there exists $N > 0$ such that if $n > N$ then $|f(x) - f_n(x)| < \epsilon/3$. In particular, $|f(x_0) - f_n(x_0)| < \epsilon/3$.

Fix $n > N$. Since f_n is continuous on S there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f_n(x) - f_n(x_0)| < \epsilon/3$.

Thus if $|x - x_0| < \delta$ and $n > N$

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \epsilon.$$

Therefore $f(x)$ is continuous on S. $\square$