Symbol Spaces and Dynamical Systems

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Honor Pledge: On my honor, I pledge that I have not given, received, or tolerated others' use of unauthorized aid in completing this work.

Signature ____

Abstract

In MCS 220 we spend almost all of our time talking about the topology of the real line and the analysis of functions on $\mathbb{R}$. These fundamental ideas generalize to higher dimensional real spaces (i.e. $\mathbb{R}^n$) and even to other topological spaces having a similar structure. In the field of dynamical systems and chaos, a very useful topological space is the *sequence space* on two symbols denoted by Σ_2. In this project you will learn about the topology of Σ_2 and the properties of a very simple, but extremely useful, function defined on this space.

1 The Basic Topology of Σ_2

We begin by defining Σ_2 which is simply the space of all possible infinite sequences of 0's and 1's. This is made precise in the following definition.

Definition 1 *The* sequence space *on two symbols is the set*

$$\Sigma_2 = \{(s_0 s_1 s_2 \cdots) \,|\, s_j = 0 \text{ or } 1\}. \tag{1}$$

In addition to the items contained in this space, we also need a way to compute the distance between items.

Definition 2 *Let* $\mathbf{s} = (s_0 s_1 s_2 \cdots)$ *and* $\mathbf{t} = (t_0 t_1 t_2 \cdots)$*. The* distance *between* $\mathbf{s}$ *and* $\mathbf{t}$ *is given by*

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}. \tag{2}$$

Note that the distance formula defined above is related very closely to the geometric series and thus I remind you that if $x \in \mathbb{R}$ and $|x| < 1$ then

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}. \tag{3}$$

Theorem 1 *If* $\mathbf{s}$, $\mathbf{t} \in \Sigma_2$ *then* $d(\mathbf{s}, \mathbf{t}) \leq 2$.

Proof: This is your first task. ■

Topological spaces that also have a way of computing the distance between elements of the space are called *metric spaces.* The distance function on such a space must satisfy the three properties given in the definition below.

Definition 3 *A function* d *is called a* metric *or* distance function *on a set* X *if for any* x, y, $z \in X$ *the following three properties hold:*

1. $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$

Theorem 2 *The distance function* d *defined in equation (2) is a metric on* Σ_2.

Proof: This is your second task. ■

We conclude this section with a theorem that tells us when two given sequences are "close" to each other.

Theorem 3 *Let* $\mathbf{s}$, $\mathbf{t} \in \Sigma_2$

1. *If* $s_i = t_i$ *for all* $i = 0, 1, \ldots, n$. *Then* $d(\mathbf{s}, \mathbf{t}) \leq 1/2^n$.
2. *If* $d(\mathbf{s}, \mathbf{t}) < 1/2^n$ *then* $s_i = t_i$ *for* $i \leq n$.

Proof: This is your third task. ■

2 Basic Properties of the Shift Map

In this section we will study the properties of a function known as the *shift map*. In particular, we will discuss some of the dynamical properties of the shift map and then conclude by showing that the shift map is a continous function.

Definition 4 *The* shift map $\sigma : \Sigma_2 \mapsto \Sigma_2$ *is defined by*

$$\sigma(s_o s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots). \tag{4}$$

In other words, the shift map simply deletes the first digit of the sequence and *shifts* the other digits left one place. For example

$$\begin{aligned} \sigma(101010\cdots) &= (010101\cdots) \\ \sigma(111111\cdots) &= (111111\cdots) \\ \sigma(100100\cdots) &= (001001\cdots). \end{aligned}$$

In dynamical systems we study what happens if we repeatedly apply the same function to some initial point. We call this process *iteration* and it generates a sequence whose convergence (or more likely non-convergence) we wish to explore. We denote the n-fold composition of a function f by f^n. In other words $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, etc. To get back to the shift map consider the point $\mathbf{s} = (101010\cdots)$. Then

$$\begin{aligned}
\sigma(\mathbf{s}) &= (010101\cdots) \\
\sigma^2(\mathbf{s}) &= \sigma(\sigma(\mathbf{s})) = (101010\cdots) \\
\sigma^3(\mathbf{s}) &= (010101\cdots)
\end{aligned}$$

and so on. In this example, we say the point $\mathbf{s}$ is of period 2 under the map σ because $\sigma^2(\mathbf{s}) = \mathbf{s}$. In general, we say that a point $\mathbf{s}$ is a period n point if $\sigma^n(\mathbf{s}) = \mathbf{s}$. This idea leads naturally to the following theorem.

Theorem 4 $\mathbf{s}$ *is a repeating sequence of length* n *(i.e.* $\mathbf{s} = (s_0 s_1 \cdots s_{n-1} s_0 s_1 \cdots)$ *) if and only if* $\sigma^n(\mathbf{s}) = \mathbf{s}$*. Moreover, if* $\mathbf{s}$ *is a repeating sequence and* $k \in \mathbb{N}$ *then* $\sigma^{kn}(\mathbf{s}) = \mathbf{s}$

Proof: This is your fourth task. ■

There is a whole lot more to say about the dynamics of the shift map but we will need a bit more topology before we can get there. Let's conclude this section with a basic analysis fact about the shift map.

Definition 5 *Suppose* $F : X \to X$ *is a function on a topological space* X *equipped with a metric* d*. Then* F *is* continuous *at* $x_0 \in X$ *if, for any* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that, if* $d(x, x_0) < \delta$ *then* $d(F(x), F(x_0)) < \epsilon$*. We say that* F *is a* continuous function *on* X *if* F *is continous for all* $x_0 \in X$*.*

Theorem 5 *The shift map* $\sigma : \Sigma_2 \to \Sigma_2$ *is continuous on* Σ_2*.*

Proof: This is your fifth task. ■

3 Advanced Properties of Σ_2 and the Shift Map

Topology is an important area of mathematics in its own right and in this course we've focused pretty extensively on the topology of the real line which is fairly intuitive. In this section I want to give you a little sneak peak of how the topological ideas that we've been discussing on $\mathbb{R}$ extend to Σ_2 and then how these topological considerations provide insight into the dynamical properties of the shift map.

Definition 6 *Let* X *be a metric space with metric* d*. If* $x_0 \in X$ *and* $r \in \mathbb{R}^+$ *then the* open disk $D_r(x_0)$ *with center* x_0 *and radius* r *is the subset of* X *defined by*

$$D_r(x_0) = \{x \in X | d(x, x_0) < r\}. \tag{5}$$

The following theorem states that in the symbol space Σ_2 open disks of radius $1/2^n$ about a point $\mathbf{s}$ contain those points $\mathbf{t}$ whose sequence agrees with $\mathbf{s}$ in the first $n+1$ entries.

Theorem 6 *Let* $\mathbf{s} = (s_0 s_1 s_2 \cdots) \in \Sigma_2$ *and let* $r = 1/2^n$ *for* $n \in \mathbb{N}$*. Then*

$$D_r(\mathbf{s}) = \{\mathbf{t} \in \Sigma_2 | \mathbf{t} = (s_0 s_1 \cdots s_n t_{n+1} \cdots)\}. \tag{6}$$

Proof: This is your sixth task. ∎

Early on in the semester when we were talking about rational numbers we mentioned that the rational numbers were *dense* in the reals (see Ross p. 24) and that this meant that although not every number is rational, the rationals are "everywhere" on the real line. Density is a very common topological idea and so we formally define it here.

Definition 7 *Let X be a metric space with metric d and let $A \subset X$. A is* dense *in X if for every $x \in X$ and $\epsilon > 0$ there exists an $a \in A$ such that $a \in D_\epsilon(x)$.*

In other words, A is dense in X if every element of X has elements of A arbitrarily close to it. This is the case of the rational numbers as a subset of the reals. On the other hand, the integers are **not** dense in the reals since there is no integer arbitrarily close to $1/2$ for example.

Theorem 7 *Let $P \subset \Sigma_2$ be the set of all periodic points of the shift map σ. Then P is dense in Σ_2.*

Proof: This is your seventh task. ∎

This denseness property is one of the defining characteristics of a *chaotic* dynamical system. One of the other fundamental properties is *sensitive dependence on initial conditions* and that is defined below.

Definition 8 *A dynamical system (or function) $F : X \to X$ displays* sensitive dependence on initial conditions *if there exists a $\beta > 0$ such that for any $x \in X$ and any $\epsilon > 0$ there exists a $y \in D_\epsilon(x)$ and a $k > 0$ such that*

$$d(F^k(x), F^k(y)) \geq \beta. \tag{7}$$

This definition is fairly complicated and it is important to understand all of the quantifiers because at first glance it almost sounds like the opposite of the definition of continuity. In *A First Course in Chaotic Dynamical Systems*, Devaney clarifies this definition as follows.

> The definition says that, no matter which x we begin with and no matter how small a region we choose about x, we can always find a y in this region whose orbit eventually separates from that of x by at least β. Moreover, the distance β is independent of x. As a consequence, for each x, there are points arbitrarily nearby whose orbits are eventually "far" from that of x.

And this leads us to the final theorem of this project.

Theorem 8 *The shift map $\sigma : \Sigma_2 \mapsto \Sigma_2$ displays sensitive dependence on initial conditions.*

Proof: This is your eighth and final task. ∎