

# MCS 357

## Discrete Dynamical Systems

Challenge Problems

February 23, 2011

Most of these problems are taken from either Devaney's graduate text *An Introduction to Chaotic Dynamical Systems, 2nd ed.* or *Dynamics and Bifurcations* by Hale and Kocak. Each problem calls for a more rigorous approach than the homework problems. Some of these problems may involve ideas that are new to you, *do not be intimidated*. With a little outside reading I'm confident that you can grasp these ideas and tackle the problem. I've put the Devaney book on reserve in the library for your use.

There are no due dates for challenge problems. You should turn in challenge problems when you think you have completed it. What you turn in should be well written, legible, etc. At the beginning of each problem you should write the problem number and statement of the problem. Proofs should be complete and examples or counter-examples clearly explained. There is no partial credit for these problems, but you will be able to redo a problem that is not correct. Have fun.

1. Consider the second-order difference equation

$$a_2x_{n+2} + a_1x_{n+1} + a_0x_n = 0$$

with  $a_i \in \mathbf{R}$  for all  $i$ .

- (a) Find a pair of linearly independent closed form solutions. In other words, find two functions  $x_n^1$  and  $x_n^2$  that satisfy this difference equation where  $x_n^1$  is not a scalar multiple of  $x_n^2$ . Note that the formulas for these solutions depend on the  $a_i$ s in some fundamental way.
  - (b) Show that if  $x_n^1$  and  $x_n^2$  are solutions then  $x_n = C^1x_n^1 + C^2x_n^2$  is also a solution.
  - (c) Find a formula for  $C^1$  and  $C^2$  in terms of initial conditions  $x_0$  and  $x_1$ .
2. Prove the following.
    - (a) Let  $\{x_n\}$  be a convergent sequence on  $\mathbf{R}$  and  $g : \mathbf{R} \rightarrow \mathbf{R}$  a continuous function. If  $\lim_{n \rightarrow \infty} x_n = L$  then  $\lim_{n \rightarrow \infty} g(x_n) = g(L)$ .
    - (b) Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be continuous and consider the sequence  $x_{n+1} = f(x_n)$ . If the sequence  $\{x_n\}$  converges to  $\bar{x}$  then  $f(\bar{x}) = \bar{x}$ .
  3. **Contraction Mapping Theorem:** A subset  $U$  of  $\mathbf{R}^n$  is *complete* if every Cauchy sequence in  $U$  converges to an element of  $U$ . A function  $f : U \mapsto U$  is a *contraction* if there exists a constant  $0 < \lambda < 1$  such that  $|f(x) - f(y)| \leq \lambda|x - y|$  for all  $x$  and  $y$  in  $U$ . Prove that if  $f$  is a contraction on a complete subset  $U$  then  $f$  has a unique (attracting) fixed point.

4. **Super Challenge:** Prove that any continuous function  $f$  from the unit ball  $B^n$  into itself must have a fixed point.

*Hint:* The unit ball  $B^n$  is the ball centered at the origin of radius 1 in  $\mathbf{R}^n$ . Thus in  $\mathbf{R}^2$  the unit ball is simply the unit disk. The boundary of  $B^n$  is  $S^{n-1}$ . In 2 dimensions this is the unit circle, in three the unit sphere etc. To prove this fixed point theorem you will need to use the following theorem.

**Theorem 1** *For all  $n > 1$  there does not exist a continuous function  $g : B^n \mapsto S^{n-1}$  such that  $g$  restricted to  $S^{n-1}$  is the identity function.*

5. Describe the phase portrait of the map  $f : S^1 \mapsto S^1$  given by

$$f(\theta) = \theta + \frac{\pi}{n} + \epsilon \sin(n\theta)$$

for  $0 < \epsilon < 1/n$ .

6. Prove that a homeomorphism of  $\mathbf{R}$  can have no periodic points with prime period greater than 2. Give an example of a homeomorphism that has a periodic point of period 2.
7. Let  $f : [a, b] \mapsto [a, b]$ . Prove that if  $f$  is continuous then  $f$  has at least one fixed point in the interval  $[a, b]$ . If, in addition,  $f$  is differentiable with  $|f'(x)| < 1$  for all  $x \in [a, b]$  prove that the fixed point is unique.
8. Prove that a homeomorphism of  $\mathbf{R}$  cannot have eventually periodic points. Is this true if the underlying space is  $\mathbf{R}^n$ ? What about  $S^1$ ?
9. Let  $f : S^1 \mapsto S^1$  be given by

$$f(\theta) = \theta + \omega + \epsilon \sin(\theta)$$

where  $\omega$  and  $\epsilon$  are constants. Prove that  $f$  is a homeomorphism of the circle if  $|\epsilon| < 1$ .

10. Let  $f : S^1 \mapsto S^1$  be given by  $f(\theta) = 2\theta$ . Prove that periodic points of  $f$  are dense in  $S^1$ .
11. Prove that eventually fixed points for the map described in the problem above are also dense in  $S^1$ .
12. Prove that the set of all periodic points of the tent map  $T(x)$  are dense in  $[0, 1]$ .
13. Let  $\Sigma'$  consist of all sequences in  $\Sigma_2$  satisfying: if  $s_j = 0$  then  $s_{j+1} = 1$ . Prove the following
- Show that the shift map  $\sigma$  preserves  $\Sigma'$  and that  $\Sigma'$  is a closed subset of  $\Sigma_2$ .
  - Show that periodic points of  $\sigma$  are dense in  $\Sigma'$ .
  - Show that there is a dense orbit in  $\Sigma'$ .

- (d) Find a recursive formula for the number of fixed points of  $\sigma^n$  in terms of the number of fixed points of  $\sigma^{n-1}$  and  $\sigma^{n-2}$ .
14. A diffeomorphism  $f : [0, 1] \mapsto [0, 1]$  is called *Morse-Smale* if  $f$  has only hyperbolic critical points. (Note that, since  $f$  is onto, the endpoints of  $[0, 1]$  are necessarily periodic.) Prove that a Morse-Smale diffeomorphism has only finitely many periodic points.