PHY 230 Midterm 1
Solution

1. *(9 pts)* Answer each of the following questions **True** or **False** and provide justification (a correct answer without justication will be worth only 1 point).

   (a) Any pair of nonzero vectors in $\mathbf{R}^2$ are linearly independent.
   Solution: False. The vectors $\vec{i}$ and $2\vec{i}$ are not linearly independent.

   (b) $y(t) = 2 + 3e^{2t}$ is a solution to the differential equation $y' = 2y - 4$.
   Solution: True. Differentiating gives $y' = 6e^{2t}$. Substituting gives $y' = 2(2 + 3e^{2t}) - 4 = 6e^{2t}$.

   (c) The vectors $\vec{v} = 3\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{w} = \vec{i} + \vec{j}$ are orthogonal.
   Solution: True. $\vec{v} \cdot \vec{w} = 0$.

2. *(6 pts)* Let $V = \left\{p(x) : p(x) = ax^2 + bx + 2b\right\}$ (i.e. V is the set of degree 2 polynomials with third coefficient twice the second).

   (a) Is $O(x) = 0$ in V? Justify.
   Solution: Yes. The polynomial $0x^2 + 0x + 0$ is in V since the third coefficient is twice the second.

   (b) If $p(x) \in V$ is $\alpha p(x) \in V$ for all scalars α? Justify.
   Solution: Yes. $\alpha p(x) = \alpha a x^2 + \alpha b x + 2\alpha b$. So $\alpha p(x)$ is a degree 2 polynomial with third coefficient twice the second.

   (c) If $p(x), q(x) \in V$ is $p(x) + q(x) \in V$? Justify.
   Solution: Yes.

3. *(10 pts)* Let

$$A = \begin{pmatrix} 3 & 11 \\ 1 & 4 \end{pmatrix}$$

and

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

   (a) Compute A^{-1} and verify your answer (i.e. compute AA^{-1}.)
   Solution: $A^{-1} = \begin{pmatrix} 4 & -11 \\ -1 & 3 \end{pmatrix}$

   (b) Use this to solve $A\vec{x} = \vec{v}$ for $\vec{x}$.
   Solution: $\vec{x} = A^{-1}\vec{v} = \begin{pmatrix} -18 \\ 5 \end{pmatrix}$

4. *(10 pts)*

$$A = \begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix}.$$

Compute the eigenvalues and eigenvectors of A. **Note:** You may use your calculator to help with arithmetic only.

Solution: The eigenvalues satisfy

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 8 = 0.$$

Using the quadratic equation we get $\lambda_1 = 2 + 2i$ and $\lambda_2 = 2 - 2i$.

To find the eigenvector $\vec{v}_1$ using λ_1 we solve

$$\begin{pmatrix} 3 - (2 + 2i) & -5 \\ 1 & 1 - (2 + 2i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for x and y. This simplifies to the single equation

$$x - (1 + 2i)y = 0.$$

Letting $y = 1$ we get $x = 1 + 2i$ and thus

$$\vec{v}_1 = \begin{pmatrix} 1 + 2i \\ 1 \end{pmatrix}.$$

Since eigenvectors come in complex conjugate pairs (for a real matrix) we get

$$\vec{v}_2 = \begin{pmatrix} 1 - 2i \\ 1 \end{pmatrix}.$$

5. *(5 pts)* The vectors $\vec{v}_1 = 1\vec{i} + 1\vec{j}$, $\vec{v}_2 = -\vec{i} + 1\vec{j} + 2\vec{k}$, and $\vec{v}_3 = 1\vec{i} - \vec{j} + 1\vec{k}$ form an *orthogonal* basis for $\mathbf{R}^3$. Thus there exists unique scalars a_1, a_2, and a_3 such that

   $\vec{i} + 2\vec{j} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3.$

   Compute a_2.

   Solution: Since this is an *orthogonal basis* we compute

   $$a_2 = \frac{\vec{w} \cdot \vec{v}_2}{v_2 \cdot \vec{v}_2} = \frac{1}{6}$$

6. *(20 pts)* Solve the following differential equations. If initial conditions are given then use them to determine the arbitrary constant(s). (Show work including integration if necessary). **Real answers only please.**

   (a) $y' + 7y = 0$ with $y(0) = -4$.
   Solution: $y = -4e^{7t}$. This is one of the 2 differential equations whose solution you should know on sight.

   (b) $y'' + 10y' + 24y = 0$
   Solution: $r^2 + 10r + 24 = (r + 6)(r + 4)$ thus $y = C_1e^{-6t} + C_2e^{-4t}$.

   (c) $y'' + 10y' + 24y = -48t + 5$
   Solution: y_h is the solution to (b). We guess $y_p = At + B$ giving

   $$24At + 10A + 24B = -48t + 5.$$

   Thus $A = -2$ and $B = 25/24$ and hence

   $$y = C_1e^{-6t} + C_2e^{-4t} - 2t + \frac{25}{24}.$$

   (d) $y'' + 4y' + 13y = 0$
   Solution: $r^2 + 4r + 13$ doesn't factor and so using the quadratic equation we get $r = -2 \pm 3i$. Thus the real solution is

   $$y = e^{-2t}\left(C_1 \cos(3t) + C_2 \sin(3t)\right).$$

7. *(15 pts)* Let

$$A = \begin{pmatrix} -1 & 4 & 3 \\ 2 & -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 7 & -4 \\ 2 & 3 \end{pmatrix},\ C = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix},\ \vec{v} = \begin{pmatrix} -7 \\ 3 \\ 0 \end{pmatrix},\ \vec{w} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

Use these matrices to answer the following questions. If the given operation is not defined then explain why.

(a) $\det(C)$ **Solution:** $\det(C) = 8$.

(b) BA **Solution:** $BA = \begin{pmatrix} -15 & 32 & 21 \\ 4 & 5 & 6 \end{pmatrix}$

(c) CA **Solution:** This is not defined since C is 3×3 and A is 2×3.

(d) CA^T **Solution:** $CA^T = \begin{pmatrix} 17 & 4 \\ 4 & 4 \\ 13 & 6 \end{pmatrix}$

(e) Is $\vec{w}$ and eigenvector of C? **Solution:**

$$C\vec{w} = \begin{pmatrix} 20 \\ 8 \\ 20 \end{pmatrix}$$

which is *not* a scalar multiple of $\vec{w}$. Hence $\vec{w}$ is not an eigenvector of C.