

## PHY 230 Midterm 3 Solution

1. (15 pts) Recall that the solution to the steady-state heat equation  $u_{xx} - u_{yy} = 0$  on a square of width  $L$  with boundary conditions  $u(0, y) = u(L, y) = u(x, 0) = 0$  is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

Find formulas for the  $A_n$  if  $L = 10$ ,

$$u(x, L) = f(x) = \begin{cases} 0 & \text{if } 0 < x < 5 \\ 5 & \text{if } 5 < x < 10 \end{cases}.$$

Please evaluate any integrals.

**Solution:** Since  $L = 10$  we use the last boundary condition to get

$$u(x, 10) = f(x) = \sum_{n=1}^{\infty} A_n \sinh(n\pi) \sin\left(\frac{n\pi x}{10}\right)$$

which is a Fourier sine series. Therefore

$$A_n \sinh(n\pi) = \frac{2}{10} \int_5^{10} 5 \sin\left(\frac{n\pi x}{10}\right) dx.$$

Evaluating the integral and solving for  $A_n$  gives

$$A_n = \frac{-10}{n\pi \sinh(n\pi)} \left( \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right).$$

2. (10 pts) The Schrödinger equation in one dimension is

$$-\frac{\hbar}{2m}\Psi_{xx} + V\Psi = i\hbar\Psi_t$$

where  $\hbar$ ,  $m$ , and  $V$  are all constant ( $i = \sqrt{-1}$ ). Use the method of separation of variables on the Schrödinger equation to find a pair of ordinary differential equations that solutions must satisfy. **Do not under any circumstances solve these ordinary differential equations.**

**Solution:** Let  $\Psi(x, t) = F(x)G(t)$  so that  $\Psi_{xx} = F''G$  and  $\Psi_t = F\dot{G}$ . Substituting into the differential equation gives

$$-\frac{\hbar}{2m}F''G + VFG = i\hbar F\dot{G}.$$

Dividing each term by  $FG$  gives

$$-\frac{\hbar}{2m}\frac{F''}{F} + V = i\hbar\frac{\dot{G}}{G} = K$$

since the left side is a function of  $x$  and the right side a function of  $t$ . These simplify to

$$\begin{aligned} -\frac{\hbar}{2m}F'' + (V - K)F &= 0 \\ i\hbar\dot{G} - KG &= 0. \end{aligned}$$

3. (15 pts) The temperature distribution of a one-dimensional wire with perfectly insulated ends is modeled by the heat equation  $u_t = c^2 u_{xx}$  with boundary conditions

$$u_x(0, t) = u_x(1, t) = 0.$$

If we assume that  $u(x, t) = F(x)G(t)$  then we get the pair of ordinary differential equations

$$F'' + \lambda^2 F = 0 \tag{1}$$

$$\dot{G} + c^2 \lambda^2 G = 0. \tag{2}$$

Show that

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi c)^2 t} \cos(n\pi x)$$

and find integral formulas for all of the coefficients  $A_n$ ,  $n = 0, 1, 2, \dots$

**Note:** Be careful with the boundary conditions as they are on the *derivative* of  $u$ .

**Solution:** The boundary conditions imply that  $F'(0) = F'(1) = 0$ .

If  $\lambda = 0$  then  $F(x) = mx + b$  and thus  $F'(x) = m$ . The boundary conditions imply that  $m = 0$  but *say nothing about  $b$* . Thus we get a solution  $F_0(x) = b_0$  where  $b_0$  is a constant.

If  $\lambda = k^2$  then

$$F(x) = A \cos(kx) + B \sin(kx)$$

and

$$F'(x) = -Ak \sin(kx) + Bk \cos(kx).$$

$F'(0) = 0$  implies that  $B = 0$ .  $F'(1) = 0$  implies that

$$-Ak \sin(k) = 0$$

and thus we get  $k_n = n\pi$  and  $F_n(x) = A_n \cos(n\pi x)$  for  $n = 1, 2, \dots$

When  $\lambda = 0$  the second differential equation becomes  $\dot{G}_0 = 0$  and thus  $G_0(t) = \text{const.}$  and consequently  $u_0(x, t) = F_0(x)G_0(t) = A_0$ . When  $\lambda_n = (n\pi)^2$  we get  $G_n(t) = e^{-(n\pi c)^2 t}$  and thus  $u_n(x, t) = F_n(x)G_n(t) = e^{-(n\pi c)^2 t} \cos(n\pi x)$ . The principle of superposition then implies that

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi c)^2 t} \cos(n\pi x).$$

4. (10 pts) The vertical displacement of a radially symmetric vibrating circular membrane of radius 1 is given by

$$u(r, t) = \sum_{n=1}^{\infty} A_n J_0(z_n r) \cos(cz_n t)$$

where  $z_n$  is the  $n$ th zero of the Bessel function  $J_0$  (i.e.  $J_0(z_n) = 0$  and  $z_n < z_{n+1}$  for all  $n$ ). Each term in this series corresponds to a unique vibrational mode. Using pictures and words, describe the vibrations of the first, second, and third modes. In particular, where are the nodal circles for each mode?

$$z_1 \approx 2.40$$

$$z_2 \approx 5.52$$

$$z_3 \approx 8.65$$

**Solution:** The three vibrational modes are

$$u_1(r, t) = A_1 J_0(z_1 r) \cos(cz_1 t)$$

$$u_2(r, t) = A_2 J_0(z_2 r) \cos(cz_2 t)$$

$$u_3(r, t) = A_3 J_0(z_3 r) \cos(cz_3 t)$$

The only value of  $r \in [0, 1]$  that makes the first mode  $u_1$  equal zero for all  $t$  is  $r = 1$ . Thus there are no nodal circles and the drum head goes up and down together.

Again if  $r = 1$  then  $u_2(1, t) = A_2 J_0(z_2) \cos(cz_2 t) = 0$ . However, there is one nodal circle for the second mode  $u_2$  when  $z_2 r = z_1$  or equivalently when  $r = z_1/z_2 \approx 0.43$ . This circle is fixed at the rim height during this vibrational mode. When the interior to this circle is “up” the exterior is “down” and vice versa.

Two nodal circles this time; one at  $r = z_1/z_3 \approx 0.28$  and the other at  $r = z_2/z_3 \approx 0.64$ . This time the vibrations are “up”, “down”, “up” and vice versa.