PHY 230 Midterm 1 Takehome Problem

Rules:

1. You may use your books and your notes.
2. Unless otherwise noted, computers or calculators may be used to evaluate integrals.
3. Discussing this exam with members of other groups is not allowed.

Honor Pledge: On my honor, I pledge that I have not given, received, or tolerated others' use of unauthorized aid in completing this work.

Signature 1: ______________________________

Signature 2: ______________________________

Signature 3: ______________________________

Print your name NEATLY on the line below.

Name 1: ______________________________

Name 2: ______________________________

Name 3: ______________________________

(25 pts) [**resonance**] In this project we will investigate the behavior of a periodically forced simple harmonic oscillator. Consider the period $2\pi/\omega$ functions

$g(t) = \sin \omega t$

and

$$f(t) = \begin{cases} -1 & \text{if } -\frac{\pi}{\omega} < x < 0 \\ 1 & \text{if } 0 < x < \frac{\pi}{\omega} \end{cases}$$

with $f(t + \frac{2\pi}{\omega}) = f(t)$

1. *(7 pts)* Show that the Fourier series representation is

   $$f(t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin((2k-1)\omega t).$$

2. *(9 pts)* Now consider the differential equation $x'' + x = g(t)$ where $g(t)$ is the function defined above. The general solution is $x(t) = x_h(t) + x_p(t)$ where x_h is the solution to $x_h'' + x_h = 0$ and x_p is called a *particular solution.*

   (a) What is $x_h(t)$ and what is its period?

   (b) Find the particular solution to this differential equation by assuming

   $x_p(t) = A\sin(\omega t).$

   (c) Using the answers from above, what is $x(t)$?

   (d) Under what conditions on ω is A undefined? Also express this conditions in terms of the period P of the forcing term. Graph solutions (for simplicity take $C_1 = 1$ and $C_2 = 0$ in x_h) to this differential equation for various values of ω as it nears the value where A is undefined. What does this say about the oscillator?

3. *(9 pts)* Now consider the differential equation $x'' + x = f(t)$ where $f(t)$ is the function defined above.

   (a) Find the particular solution to this differential equation by assuming

   $$x_p(t) = \sum_{n=1}^{\infty} \alpha_k \sin((2k-1)\omega t).$$

   and then finding an expression for each α_k.

   (b) Under what conditions on ω does there exist a k such that α_k is undefined? Also express this conditions in terms of the period P of the forcing term. What are the physical implications of this?