

## PHY 230 Midterm 1 Takehome Problem

**Rules:**

1. You may use your books and your notes.
2. Unless otherwise noted, computers or calculators may be used to evaluate integrals.
3. Discussing this exam with members of other groups is not allowed.

**Honor Pledge:** On my honor, I pledge that I have not given, received, or tolerated others' use of unauthorized aid in completing this work.

Signature 1: \_\_\_\_\_

Signature 2: \_\_\_\_\_

Print your name NEATLY on the line below.

Name 1: \_\_\_\_\_

Name 2: \_\_\_\_\_

# Bungee Jumping

Suppose that you have no sense. Suppose that you are standing on a bridge above the Malad River canyon. Suppose that you plan to jump off that bridge. You have no suicide wish. Instead, you plan to attach a bungee cord to your feet, to dive gracefully into the void, and to be pulled back gently by the cord before you hit the river that is 53 meters below. You have brought several different cords with which to affix your feet, including several standard bungee cords, a climbing rope, and a steel cable. You need to choose the stiffness and length of the cord so as to avoid the unpleasantness associated with an unexpected water landing. You are undaunted by this task, because you know math!

Each of the cords you have brought will be tied off so as to be 30 meters long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that "natural length" as  $x(t)$ , where  $x$  increases as you go down and is a function of time  $t$ . Then at the time you jump,  $x(0) = -30$ , while if your 2 meter frame hits the water head first, then at that time  $x(t) = 53 - 30 - 2$ .

You know that the acceleration due to gravity is a constant, called  $g$ , so that the force pulling downwards on your body is  $mg$ . You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in the opposite direction to your motion of about  $av$ , where  $a$  is a constant and  $v$  is your velocity. Finally, you know that Hooke's law describing the action of springs says that the bungee cord will eventually exert a force on you proportional to its distance past its natural length. Thus, you know that the force of the cord pulling you back from destruction may be expressed as


$$b(x) = \begin{cases} 0 & x \leq 0 \\ -kx & x > 0 \end{cases}$$


The number  $k$  is called the *spring constant*, and is where the stiffness of the cord you use influences the equation. For example, if you used the steel cable, then  $k$  would be very large, giving a tremendous stopping force very suddenly as you passed the natural length of the cable. This could lead to discomfort, injury, or even a Darwin award. You want to choose the cord with a  $k$  value large enough to stop you above or just touching the water, but not too suddenly. Consequently, you are interested in finding the distance you fall below the natural length of the cord as a function of the spring constant. To do that, you must solve the differential equation that we have derived in words above: the force  $mx''$  on your body is given by


$$mx'' = mg + b(x) - ax'.$$

Here  $m$  is your mass, 72 kg, and  $x'$  is the rate of change of your position below the equilibrium with respect to time; i.e. your velocity. The constant  $a$  for air resistance depends on a number of things, including whether you wear your skin-tight pink spandex or your skater shorts and XXL T-shirt, but you know that the value today is 2.8.


This is a nonlinear differential equation, unlike any we have seen before. However, inside this nonlinear equation are two linear equations, struggling to get out. When  $x < 0$ , the equation is  $mx'' = mg - ax'$ , while after you pass the natural length of the cord it is  $mx'' = mg - kx - ax'$ . We will solve these separately, and then piece the solutions together when  $x(t) = 0$ .

**Exercise 1**  Solve the equation  $mx'' + ax' = mg$  for  $x(t)$ , given that you step off the bridge - no jumping, no diving! Stepping off means  $x(0) = -30$ ,  $x'(0) = 0$ . You may use  $m = 72$ ,  $a = 2.8$ , and  $g = 9.8$ .

**Exercise 2**  Use the solution from Exercise 1 to compute the length of time you freefall (the time it takes to go the natural length of the cord: 30 meters).

**Exercise 3**  Compute the derivative of the solution you found in Exercise 1 and evaluate it at the time you found in Exercise 2. You have found your downward speed when you pass the point where the cord starts to pull.


Exercise 1 has given you an expression for your position  $t$  seconds after you step off the bridge, before the bungee cord starts to pull you back. Notice that it does not depend on the value for  $k$ . When you pass the natural length of the bungee cord, it does start to pull back, so the differential equation changes. Let  $t_1$  denote the time you computed in Exercise 2, and let  $v_1$  denote the speed you calculated in Exercise 3.


**Exercise 4**  Solve the initial value problem

$$\begin{aligned} mx'' + ax' + kx &= mg; \\ x(t_1) &= 0; \\ x'(t_1) &= v_1. \end{aligned}$$

For now you may use the value  $k = 200$ , but eventually you will need to replace that with the actual values for the cords you brought. The solution  $x(t)$  represents your position below the natural length of the cord after it starts to pull back.


Now we have an expression for our position as the cord is pulling on us. All we have to do is to find out the time  $t_2$  when we stop going down. When we stop going down, our velocity is zero, i.e.  $x'(t_2) = 0$ .


**Exercise 5**  Compute the derivative of the expression you found in Exercise 4 and solve for the value of  $t$  where it is zero. This time is  $t_2$ . Be careful that the time you compute is greater than  $t_1$  - there are several times when your motion stops at the top and bottom of your bounces! After you find  $t_2$ , substitute it back into the solution you found in Exercise 4 to find your lowest position.


**Exercise 6**  You have brought a soft bungee cord with  $k = 102$ , a stiffer cord with  $k = 196$ , and a climbing rope for which  $k = 284$ . Which, if any, of these may you use safely under the conditions given?


As you can see, knowing a little bit of math is a dangerous thing. We remind you that the assumption that the drag due to air resistance is linear applies only for low speeds. By the time you swoop past the natural length of the cord, that approximation is only wishful thinking, so your actual mileage may vary. Moreover, springs behave nonlinearly in large oscillations, so Hooke's law is only an approximation. Do not trust your life to an approximation made by a man who has been dead for two hundred years. Leave bungee jumping to the professionals.

### Still Curious?

**Exercise 1**  You have a bungee cord for which you have not determined the spring constant. To do so, you suspend a mass of 5 kg from the end of the thirty meter cord, causing the cord to stretch 0.35m. What is the  $k$  value for this cord? You may neglect the mass of the cord itself.

**Exercise 2**  If your friend uses the bungee cord with  $k = 196$ , what happens? Your friend has a mass of 102kg.

**Exercise 3**  If your heavy friend wants to jump anyway, how short should you make the cord so that he does not get wet?

**Exercise 4**  Use a computer algebra system to graph the solution you found in Exercise 1 and the one from Exercise 4 on the same axes. Explain the differences.