



Physics 230 Takehome III

A Vibrating Square Membrane


Rules and guidelines: This takehome is worth 50 points toward the 100 point exam total. It is due on **Friday, May 15 in class.**

1. You are to work in your assigned groups. Talking about this project with members of other groups is not permitted. Asking advice from other students and/or faculty members is not permitted.
2. The use of your text or other written resources is permitted. Please cite these works appropriately.
3. The  icon indicates that the problem should be done primarily by hand. Calculators and or computers may be used for integrals and other basic calculations.
4. The  icon indicates that a graphing calculator or computer is necessary for that problem.
5. Turn in one solution per group. It need not be typed but it should be legible. Include comments with your calculation to make your work understandable.

In this project you will consider a model of a vibrating square membrane (a square drum). The partial differential equation modeling the vertical displacement of this membrane is


$$u_{tt} = c^2 (u_{xx} + u_{yy}). \quad (1)$$

We will assume that the drum is a square having length and width equal to π . The edges of the drum are fixed at a reference height of zero and therefore $u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0$.

Exercise 1  (6 pts) Assume that $u(x, t) = F(x)G(y)H(t)$ and derive the differential equations


$$F''G + FG'' + K_1^2 FG = 0 \quad (2)$$


$$\ddot{H} + c^2 K_1^2 H = 0 \quad (3)$$


Exercise 2  (5 pts) Show that the first equation above can be further reduced to the pair of ordinary differential equations

$$F'' + K_2^2 F = 0 \quad (4)$$

$$G'' + (K_1^2 - K_2^2)G = 0 \quad (5)$$


Exercise 3  (5 pts) Solve (4) subject to the boundary conditions above. In particular, explain why $K_2 = m$ where $m = 1, 2, \dots$ and $F_m(x) = \sin(mx)$.

Exercise 4  (5 pts) Solve (5) subject to the boundary conditions above. In particular, explain why $K_1^2 - K_2^2 = n^2$ where $n = 1, 2, \dots$ and $G_{mn}(y) = \sin(ny)$.

Exercise 5  (6 pts) Solve (3). In particular, explain why

$$H_{mn}(t) = A_{mn} \cos(\lambda_{mn}t) + B_{mn} \sin(\lambda_{mn}t)$$


where $\lambda_{mn} = c\sqrt{m^2 + n^2}$.

Exercise 6  (6 pts) What is u_{mn} ? Describe the vibrational modes of u_{11} , u_{12} , u_{21} , and u_{22} .

Using the principle of superposition, we know that the sum of solutions is again a solution. Therefore the general solution to this differential equation is a double summation

$$u = \sum_{m=1} \sum_{n=1} [A_{mn} \cos(\lambda_{mn}t) + B_{mn} \sin(\lambda_{mn}t)] \sin(m\pi x) \sin(n\pi y). \quad (6)$$


This is called a double Fourier series. The goal of the next few steps is to find a formula for the coefficients. We will assume initial conditions of $u(x, y, 0) = f(x, y)$. and $u_t(x, y, 0) = 0$.

Exercise 7  (5 pts) Show that the derivative condition implies that $B_{mn} = 0$ for all m and n .

Exercise 8  (6 pts) Let

$$\alpha_m = \sum_{n=1} A_{mn} \sin(n\pi y). \quad (7)$$

Write a Fourier series for $f(x, y)$ having coefficients α_m . Use the Fourier coefficient formula to express α_m as an integral.

Exercise 9  (6 pts) Use equation 7 to express A_{mn} as a double integral.