PHY 230 Midterm 3
Takehome

Rules: There is no time limit on this portion of the midterm. You are to work in groups of 2 and turn in 1 solution per group. Your solution should be written *neatly* with relevant calculations provided and brief explanations where necessary. You are free to use any resource available to you (books, computers, etc.) other than faculty members and other students. You must cite these references explicitly. This portion of the exam is due by Wednesday, May 14 in class.

Honor Pledge: On my honor, I pledge that I have not given, received, or tolerated others' use of unauthorized aid in completing this work.

Signature 1: ______________________________

Signature 2: ______________________________

Print your name NEATLY on the line below.

Name 1: ______________________________

Name 2: ______________________________

Laplace's Equation in Spherical Coordinates Recall that Laplace's equation $\nabla^2 u = 0$ arises in a variety of applications including steady-state heat flow, fluid flow, electrostatic potential, and gravitational potential. In this project we will be solving Laplace's equation in spherical coordinates. For the sake of definiteness let's assume that we have a sphere of radius 1 that has a given temperature distribution and we want to compute the temperature at any point in space.

Recall that given a point P with cartesian coordinates (x, y, z), spherical coordinates are defined by

$$x = r\cos\theta\sin\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\phi$$

where $r^2 = x^2 + y^2 + z^2$ is the distance from the origin, θ is the angle made with the positive x-axis and the point $(x, y, 0)$ and ϕ is the angle made with the positive z-axis. Laplaces equation in spherical coordinates is

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right] = 0. \tag{1}$$

1. *(4 pts)* Suppose on the sphere $r = 1$ centered at the origin the temperature $u(1, \theta, \phi) = f(\phi)$ (in other words the temperature is independent of θ or equivalently the temperature is constant along lines of latitude.) Explain why Laplace's equation can be simplified to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) = 0. \tag{2}$$

2. *(4 pts)* In addition to the "boundary condition" on the unit sphere given above we also require two additional conditions

$$\begin{aligned} \lim_{r\to\infty} u(r,\phi) &= 0 \qquad (3) \\ u(0,\phi) &= 0. \qquad (4) \end{aligned}$$

Explain why these assumptions are physically relevant.

3. *(6 pts)* Use separation of variable $u(r, \phi) = G(r)H(\phi)$ to reduce equation (2) to the pair of ordinary differential equations

$$\begin{aligned} \frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH &= 0 \qquad (5) \\ \frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) &= k \qquad (6) \end{aligned}$$

where k is an arbitrary constant.

4. *(7 pts)* Let's solve equation (6) first. Let $k = n(n+1)$ and show that (6) becomes

$$r^2 G'' + 2rG' - n(n+1)G = 0.$$

Show that two linearly independent solutions of this differential equation are $G_n(r) = r^n$ and $G_n^*(r) = r^{-n-1}$.

5. *(10 pts)* Now let's solve the other equation. Let $\cos\phi = w$ and use this to show that

$$\frac{d}{d\phi} = -\sin\phi \frac{d}{dw}.$$

Use this and the fact that $k = n(n+1)$ to rewrite (5) as

$$(1-w^2)\frac{d^2 H_n}{dw^2} - 2w\frac{dH_n}{dw} + n(n+1)H_n = 0.$$

6. *(5 pts)* Note that this is Legendre's Equation. Explain why the solution to equation (5) is thus $H(\phi) = P_n(\cos\phi)$.

7. *(4 pts)* Explain why each of the functions $u_n(r,\phi) = A_n r^n P_n(\cos\phi)$ is a solution to equation (2) *inside* the unit sphere while each of the functions $u_n^*(r,\phi) = B_n r^{-n-1} P_n(\cos\phi)$ are solutions *outside* the unit sphere.

8. *(10 pts)* Let's restrict our attention to solutions inside the sphere only. By the principal of superposition

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi)$$

is the general solution to the partial differential equation in this region. Find a formula for each of the coefficients A_n. Justify your answer.