David Wolfe
Proofs by Induction
November 18, 2002

1 Top-down induction

1. To prove a statement $P(n)$ for all $n \in S$, feel free at any time (i.e., whenever convenient) to assume $P(m)$ is true for $m < n$. Whenever such an assumption is invoked write, "by induction."
2. Carefully review your proof, and prove any case not handled — "base case."

A few notes are in order:

- You may be concerned that this form of induction seems to be going "backwards". Rather than saying "Assume $P(n)$ and prove $P(n+1)$" as you may have first learned, this says "Prove $P(n)$ by using $P(m)$ for $m < n$. The two views are totally consistent; both use the smaller cases to help prove larger cases.
- You never have to decide in advance that you plan to use induction. You use it whenever you wish in *any* proof, and later worry about any base cases you may have missed.
- S can be any set of points (or integers or vectors or ...) and $<$ can be any partial ordering for which each element in S has "finite height." I.e., each element $n \in S$ has a height, $h(n)$, so that there's *no* sequence of $h(n)+1$ points starting at n each less than the previous:

  $$n = n_0 > n_1 > n_2 > \cdots > n_{h(n)}$$

  The most common example is non-negative integers under the usual $<$. For most proofs in computer science or discrete mathematics, these requirements are a mere technicality, which will not be addressed in these notes.

2 Examples

Example 1 *Show that the sum of the first* n *positive integers is* $n(n+1)/2$.

Proof:

$$\begin{aligned} 1+2+\cdots+n &= (1+2+\cdots+(n-1))+n && (1)\\ &= (n-1)n/2+n, \text{ (by induction)} && (2)\\ &= n(n+1)/2 && (3) \end{aligned}$$

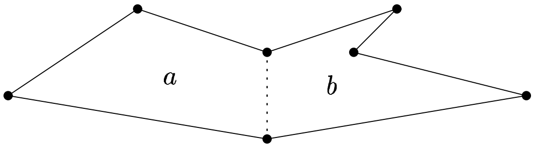
The base case is $n = 0$, and $n(n+1)/2 = 0$ as it should. ■

Note that the base case of $n = 0$ is the **only** case not handled by (1). In particular, when $n = 0$, there are no terms in $1+2+\cdots+n$, and so there's no n to separate out.

When trying to solve any problem, first try to break up your problem into smaller pieces. In Equation (1) above, there was one very natural way to do this, and induction followed naturally. The next two examples illustrate this even more clearly.

Example 2 *Show that the sum of the interior angles of an n-sided polygon is* $180 \cdot (n-2)$.

Proof: One can always find a line segment which separates the n-gon into two smaller polygons:[1]



Call the number of sides in each a and b each less than n. The sum of the angles of the two smaller polygons are $180 \cdot (a-2)$ and $180 \cdot (b-2)$, by induction. Now, each interior angle of the original polygon is either an interior angle of the two smaller polygons, or it's the sum of two angles, one from each of the smaller polygons. So the sum of the angles of the original polygon is exactly the sum of the angles of the two smaller polygons. Also, $a+b=n+2$ since adding the line segment added two sides, one two each smaller polygon. So, $180 \cdot (a-2) + 180 \cdot (b-2) = 180 \cdot (a+b-4) = 180 \cdot (n-2)$. The base case is a single triangle with angles summing to 180. ∎

The base case is chosen to be a single triangle because that's when the induction argument fails; there's no way to break up a triangle into two smaller polygons.

Example 3 *Let the Fibonacci numbers,* a_n *be given by:*

$$a_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ a_{n-1} + a_{n-2} & \text{if } n \geq 2 \end{cases}$$

Show $a_n = a_r a_{n-r-1} + a_{r+1} a_{n-r}$ *for* $0 \leq r < n$

Proof:

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} && (4) \\ &= (a_r a_{n-r-2} + a_{r+1} a_{n-r-1}) + (a_r a_{n-r-3} + a_{r+1} a_{n-r-2}), \text{ by induction} && (5) \\ &= a_r (a_{n-r-2} + a_{n-r-3}) + a_{r+1} (a_{n-r-1} + a_{n-r-2}) && (6) \\ &= a_r a_{n-r-1} + a_{r+1} a_{n-r}, \text{ by the definition of Fibonacci numbers.} && (7) \end{aligned}$$

Now, to figure out the base cases we need to see how the argument above can fail. (4) fails when $n = 0$ or $n = 1$. (5) fails when $r = n-1$ or $r = n-2$, since the statement of the theorem insists that $r < n$. In other words, when invoking the induction hypothesis, we need $r < n-1$ to expand a_{n-1} and $r < n-2$ to expand a_{n-2}. (6) and (7) introduce no further problematic cases. So the base cases are:

- $n = 0$: No $0 \leq r < n$ exists, so the theorem is vacuously true.
- $n = 1$: Then $r = 0$ and $a_0 a_0 + a_1 a_1 = 0 + 1 = 1 = a_1$.
- $r = n-1$: $a_{n-1} a_0 + a_n a_1 = a_n$

---

[1] This is actually a subtle point, whose proof is tangential to the points I'm trying to make. I'll sketch one approach. The polygon will have some acute angle, say $\angle ABC$. If line segment $\overline{BC}$ does not intersect the polygon, we are done. Otherwise, find the vertex, P, nearest B in $\triangle ABC$. Adding the segment $\overline{BP}$ divides the polygon into two polygons.

- $r = n-2$: $a_{n-2}a_1 + a_{n-1}a_2 = a_{n-2} + a_{n-1} = a_n$

■

Note that there's really no need to differentiate between "base cases" of an induction and "special cases" of any proof. One could rewrite the whole proof without any "base case", and the choice is a matter of style:

Proof: (Alternate proof sketch.) If $n = 0$ or $n = 1$, we can verify the theorem holds. Otherwise,

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2},\ n \geq 2 \\ &= \begin{cases} a_{n-1}a_0 + a_n a_1 = a_n & \text{if } r = n-1 \\ a_{n-2}a_1 + a_{n-1}a_2 = a_{n-2} + a_{n-1} = a_n & \text{if } r = n-2 \\ (a_r a_{n-r-2} + a_{r+1}a_{n-r-1}) + (a_r a_{n-r-3} + a_{r+1}a_{n-r-2}) & \text{otherwise (by induction)} \\ \quad = a_r(a_{n-r-2} + a_{n-r-3}) + a_{r+1}(a_{n-r-1} + a_{n-r-2}) & \\ \quad = a_r a_{n-r-1} + a_{r+1}a_{n-r} & \end{cases} \end{aligned}$$

■

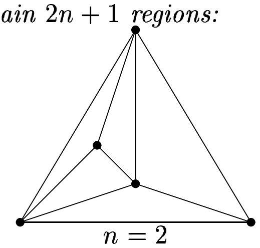
3 Why is top-down induction better?

The usual axiomatic presentation of mathematical induction suggests that to prove some statement $P(n)$ for $n \geq 0$,

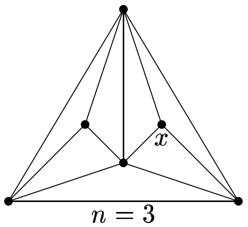
1. Prove the base case, $P(0)$
2. Induction step: Assume $P(n)$ and prove $P(n+1)$

There are many reasons why top-down induction is better to use than the axiomatic approach. The most important is that in complicated proofs using axiomatic induction it's easy to overlook some cases of size $n+1$. This is a popular pitfall among even the best of mathematicians! One such false proof will exemplify the pitfall.

Example 4 *Prove that n internal points triangulate a triangle into $2n+1$ smaller triangles. In other words, n points are placed in the interior of a triangle indicated by 3 points. If non-crossing line segments (edges) are introduced connecting the $n+3$ points until no more can be added, then the triangle will contain $2n+1$ regions:*



$n = 2$



$n = 3$

Proof: *(False proof)* For a base case, if there are $n = 0$ added points, there are $2n+1 = 1$ triangles. For the inductive step, assume that if there are n points, there must be $2n+1$ triangles. To prove the case for $n+1$, add one point more point, say x, to the triangle. This removes 1 region, but creates 3, leaving a total of $2n+1-1+3 = 2(n+1)+1$ regions. ■

This argument is very seductive; in fact most are taken in by it unless warned it's fallacious. The reason the proof is incomplete is that we have not argued about *all* possible triangulations — only those that can be arrived at from smaller triangulations. You may be tempted to patch the proof by permitting the addition of points along previous edges, but that still would not be good enough! With some work, you can come up with a triangulation on $n = 11$ interior points where each point is incident to at least 5 edges. This triangulation could not have been arrived at from a smaller triangulation with the addition of one point — the last point added would only be incident to 3 edges (or 4 edges in the proposed "patch").

Proof: (Correct proof of Example 4) Begin with a triangulation on n interior points. Remove one of the points. If the point had k incident edges, then k triangles will be removed, leaving a k-sided polygonal region. A k-sided polygon can be divided into $k-2$ triangles (by an argument parallel to Example 2). By induction, this new triangulation has $2n-1$ triangles, but it has $k-(k-2)$ or 2 fewer triangles than the original. So the original triangulation had $2n+1$ triangles. The base case of $n = 0$ is trivial. ■

Being able to write consistently correct proofs is the most important advantage with top-down induction. But a few more advantages are worth noting:

- The axiomatic model's assumption that you're always trying to prove a statement $P(n)$ for $n \geq 0$ is simplistic. First, in actual proofs, the parameter n might take on many different forms. Further, determining what in your proof "should" serve the role of n is not always obvious. Of course, those investigating the fundamentals of mathematics would consider this simplistic view a benefit, but I think most students are interested in understanding how to apply induction in the widest possible context.

  Top-down induction can be proved using the axiomatic form, so it's really no more general. (A good exercise — Induct on the height, $h(n)$.) But the top-down form is much easier to use in wider contexts.

- In a complicated proof, it's unclear what is the "right" base case to choose, particularly if the induction step hasn't been proven yet. In Example 3, it's hard to foresee what base case is needed before doing the induction argument; so why do the base case first? Further, top-down induction gives guidelines for how to choose the strongest possible, minimal sized base case.

- Students often wonder, "How do I know when to use induction?". This is a surprisingly hard question to answer. The top-down induction paradigm sidesteps this issue somewhat, since there is no need for the prover to know in advance that induction is required. Through the process of breaking up a problem or expanding or massaging an expression, if a smaller case ever appears, induction can be used freely.

- Induction on several variables is completely demystified. The prover need not know in advance what parameters induction will be used on, nor whether induction will be used on several variables.

- The axiomatic induction suggests that to prove $S(n)$, first assume (by way of induction) $S(n)$. This *sounds* circular. Of course it isn't, since you'd then proceed to prove $S(n+1)$. But since you're trying to prove $S(n)$ doesn't it make more sense to assume $S(n-1)$ and prove $S(n)$? Top-down induction just goes one step further, allowing you to assume $S(m)$ for $m < n$.