In-class exercises

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Important Saturdays to mark on your calendar: November 9 is the ACM programming contest. November 16 is the PennePutnam team contest on-site. December 7 is the Putnam. February 22 is the Konhauser. The Mathematical Contest in Modeling will be held during a weekend in early February.

For this week's problems, you should not come up with a complete solution (unless you are able). Instead, work on *getting your hands dirty,* playing with the problem for long enough that you can make a reasonable conjecture. Often if a problem has a very large parameter, working with smaller examples will help!

In short, these highlight the problem solving techniques of *wishful thinking* and *make it easier* which we discussed last week, but are more mathematical and less puzzle-like.

1. (Zeitz 2.2.3) Lockers are numbered $1, 2, 3, \ldots, 1000$. At first all lockers are closed. A person walks by, and opens every other locker, starting with number 2. Thus lockers $2, 4, 6, \ldots, 1000$ are open. Another person walks by, and changes the "state" (i.e., closes a locker if it is open, opens a locker if it is closed) of every third locker, staring with number 3. Then another person changes the state of every 4th locker starting with number 4, etc. This process continues for 1000 people. Which lockers will be closed at the end?

2. (Zeitz 2.2.2, AIME 1985) The numbers in the sequence

   $$101, 104, 109, 116, \ldots$$

   are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum values of d_n as n ranges through the positive integers.

3. (Zeitz 2.2.9) Define $f(x) = 1/(1-x)$ and denote r iterations of the function f by f^r. Compute $f^{1999}(2000)$.

4. (Zeitz 2.2.11) Let N denote the natural numbers $\{1, 2, 3, 4, \ldots\}$. Consider the function $f : N \longrightarrow N$ which satisfies $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1$ for all $n \in N$. Find a nice simple algorithm for computing $f(n)$. Your algorithm should be a short single sentence long.

5. (Zeitz 2.2.15) For each integer $n > 1$, find *distinct* positive integer x and y such that

   $$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$