MCS-200
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In-class exercises

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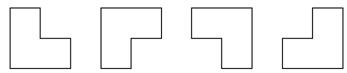
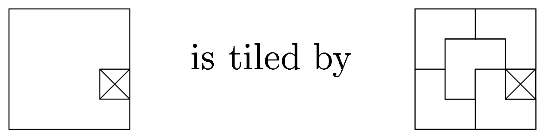
This week's focus is on inductive reasoning.

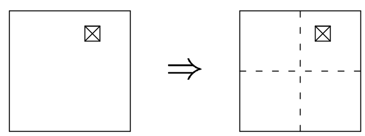
1. A *triomino* is a shape consisting 3 squares in an L-shape in any orientation. Here are the four possible orientations of a triomino:



Prove that a $2^n \times 2^n$ checkerboard with **any** square removed can be *tiled* with triominoes. For example,



(A tiling must cover each square exactly once; no two triominos may overlap.) Picture hint:



2. Into how many regions do n great circles (no three intersecting at one point) decompose the surface of the sphere in which they lie? Great circles are the largest circles around the sphere (such as an equator or a circle through both poles). Be sure to prove your answer.

3. (Larson 1.1.3) Let $x_1, x_2, \ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}, \qquad n = 3, 4, 5, \ldots$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

Hint: First find a formula for x_n in terms of x_1 and x_2 and prove that formula holds by induction.

4. Define the fibonacci numbers, $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$. Prove that $f_{2n+1} = f_{n+1}^2 + f_n^2$. (Hint: you'll need to introduce an additional induction hypothesis about f_{2n}.)

5. (Manber 2.23 — Pick's Theorem) The *lattice points* in the plane are the points with integer coordinates. Let P be a polygon that does not cross itself (i.e., a *simple* polygon) such that all of its vertices are lattice points. Let p be the number of lattice points that are on the boundary of the polygon (including its vertices), and let q be the number of lattice points that are inside. Prove that the area of the polygon is $p/2 + q - 1$.

Hint: First prove that if P_1 is the union of polygons P_2 and P_3, then if two of the three polygons satsify Pick's Theorem, so does the third. Then, use inductive reasoning, but you don't need reduce to *smaller* polygons, just simpler ones.