In-class exercises

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Larson identifies 12 heuristics for attacking problems:

1. Search for a pattern
2. Draw a figure
3. Formulate an equivalent problem
4. Modify the problem
5. Choose an effective notation
6. Exploit symmetry
7. Divide into cases
8. Work backwards
9. Argue by contradiction
10. Pursue parity
11. Consider extreme cases
12. Generalize

This week I'll include one problem which exemplifies each of the first size approaches:

1. (Larson 1.1.3) Let $x_1, x_2, \ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}, \qquad n = 3, 4, 5, \ldots$$

   Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

2. (Larson 1.2.2) A particle moving on a straight line starts from rest and attains a velocity v_0 after traversing a distance of s_0. If the motion is such that the acceleration was never increasing, find the maximum time for the transverse.

3. (Larson 1.3.8) Show that $x^7 - 2x^5 + 10x^2 - 1$ has no root greater than 1. (Hint: It's generally easier to show that an equation has no *positive* roots, so reformulate this into an equivalent problem.)

4. (Larson 1.4.4) Evaluate $\int_0^\infty e^{-x^2}$. (Hint: It's easier to square the expression and evaluate the double integral, even though that looks like a harder problem!)

5. (Larson 1.5.1) One morning it started snowing at a heavy and constant rate. A snowplow started out at 8:00am. At 9:00am it had gone 2 miles. By 10am it had gone 3 miles. Assuming that the snowplow removes a constant volume of snow per hour, determine the time at which it started snowing.

6. (Larson 1.6.7) Fifteen pennies are arranged as below. Each penny is either heads up or tails up. Prove that there exist three pennies with the same side up whose centers are the vertices of an equilateral triangle. (There are many ways to exploit symmeter and create "without loss of generality" arguments.)