MCS-200 **Putnam practice** December 2, 2002

David Wolfe

1. (Putnam 1998 A-4) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by the concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_2 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

2. (Putnam 1998 A-2) Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

3. (Putnam 1984 A-5) Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leq 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral
$$\iiint_R x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$
in the form $a!b!c!d!/n!$, where a, b, c, d and n are positive integers.

4. (Putnam 1984 B-1) Let n be a positive integer, and define
$$f(n) = 1! + 2! + 3! + \cdots + n!$$
Find polynomials $P(x)$ and $Q(x)$ such that
$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$
for all $n \geq 1$.

5. (Putnam 1984 B-3) Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F,

   (a) $x * z = y * z$ implies $x = y$ (right cancellation holds)

   (b) $x * (y * z) \neq (x * y) * z$ (*no* case of associativity holds)